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A VARIATIONAL SETTING FOR VOLUME CONSTRAINED IMAGE REGISTRATION

Christiane Pöschl

Departament de Tecnologia, Universitat Pompeu Fabra
Edificio Tánger, Tánger 122-140, 08018 Barcelona, Spain

Jan Modersitzki

Institute of Mathematics and Image Computing, Universität Lübeck
Wallstr. 40, 23560 Lübeck, Germany

Otmar Scherzer

Computational Science Center, University of Vienna
Nordbergstrae 15, 1090 Wien, Austria
RICAM, Austrian Academy of Sciences
Altenberger Straße 69, 4040 Linz, Austria

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Abstract. We consider image registration, which is the determination of a geometrical transformation between two data sets. In this paper we propose constrained variational methods which aim for controlling the change of area or volume under registration transformation. We prove an existence result, convergence of a finite element method, and present a simple numerical example for volume-preserving registration.

1. **Introduction.**

Registration is concerned with the determination of a geometrical transformation that aligns points in one view of an object with corresponding points in another view of the same or a similar object. There exist many applications, particularly in medical imaging, which demand for registration. Examples include the treatment verification of pre- and post-intervention images, the study of temporal series of cardiac images, and the monitoring of the time evolution of a contrast agent injection subject to patient motion. Another important application is the combination of information from multiple images, acquired using different modalities such as computer tomography (CT) and magnetic resonance imaging (MRI), a technique also known as *fusion*. In the last two decades, computerized non-rigid image registration has played an increasingly important role in medical imaging, see, e.g., [21, 27, 12, 35, 24] and references therein.

Recent work on image registration concerns taking into account prior information on the geometrical transformation. For example, many applications require the transformation to be one-to-one. In this context two major directions have been suggested. One approach facilitates diffeomorphic or geodesic splines; see, e.g.,

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[32, 9, 4, 23, 22, 33, 19]. The underlying idea is to add time as a further dimension and to establish an energy minimizing flow of correspondent particles. An additional regularization enforces that particles can not cross and as a consequence, the flow and hence the transformation is one-to-one. These techniques are of particular interest for constructing transformation groups and for performing shape analysis. However, these techniques require an additional time integration and do not provide full control on the transformation. The second approach is based on constrained optimization. Introducing the displacement u with $y(x) = x + u(x)$, the constraints are based on the determinant of the Jacobian

$$C(u)(x) = \det\left(\nabla(x + u(x))\right) \quad \text{for almost all} \quad x\ .$$

Equality $C(u)(x) = 1$ as well as inequality $k(x) \leq C(u)(x) \leq K(x)$ constraints have been discussed in the literature [17, 18]. The choices $0 < k(x) \leq K(x) < \infty$ provide lower and upper bounds for volume changes. In contrast to the diffeomorphic approach, the equality constrained approach guarantees that the volume of tissue is constant under transformation. The inequality approach enables the usage of pre-knowledge. For example, one could restrict on subregions displaying bones with $k(x) = K(x) = 1$ and on subregions displaying soft-tissue with some relaxed bounds. It is worthwhile noticing, that there is a connection to local rigidity constraints, where with I_d the identity matrix in $\mathbb{R}^d$, the constraints are essentially

$$C^{\text{rigid}}(u)(x) = \nabla u^\top(x)\nabla u(x) - I_d = 0.$$

Volume preservation and local rigidity have been treated in terms of soft-constraints and hard constraints. For soft-constraints, an appropriate penalty term is added to the overall objective [30, 31, 25, 34], whereas hard constraints are much more delicate. For the local rigidity approach, it has been shown that a Lagrangian framework leads to linear constraints and thus enables an efficient numerical implementation and analysis [15]. Numerical schemes have been proposed for the non-linear and differential volume preserving hard constraints [17, 18]. However, existence theory has only be established for a finite dimension setting.

In this paper, we use variational techniques to derive an existence theory for a minimizing element of the constrained optimization problem in an infinite dimensional setting. From a mathematical point of view the work most closely related work is [8], where variational regularization methods motivated from nonlinear elasticity have been used. However, there the minimization problem is treated in an unconstrained setting.

This paper is organized as follows. In Section 2 we introduce the registration setup for this paper, then we quote some important results from the calculus of variation, that are relevant for this work. Using them we prove existence of minimizers of the constrained regularization functionals for registration in Section 4. Theorem 4.1 indicates that in the volume constrained case, the spacial dimension determines the choice of the regularization functional strongly. A central part of this work is the convergence analysis of the finite dimensional approximation of the minimization problem. In Section 5 we explain how to approximate the constraints and the involved functionals, and show in Theorem 5.5 that under certain conditions, the approximated regularized solution converges to a solution of the original registration problem. At the end we give a brief outline how we implemented the area/volume preserving image registration problem and show a simple numerical result.

2. **The registration setup.** Given a reference image I_0 and a template image I_1, which are assumed to be compactly supported functions $I_0, I_1 : \Omega \to \mathbb{R}$, where $\Omega =]0,1[^d$ and $d = 2$ or $d = 3$. Hence, $I_1(x)$ is a gray value at spatial position x and $I_0(x) = I_1(x) = 0$ for all $x \notin \Omega$. The objective is to find a displacement $u = (u^1, \dots, u^d) : \Omega \to \mathbb{R}^d$ such that the distance between the transformed template image $I_1(id + u)$ and the reference is small in an appropriate sense. In principle every integral based distance measure can be used; see, e.g., [29, 20, 24] for an overview. The most widely used distance measure is the squared distance

$$\mathcal{S}(u) = \int_\Omega |I_1(x + u(x)) - I_0(x)|^2 \, dx \,. \tag{1}$$

Minimizing the distance measure is known to be ill-posed, in the sense that the minimizers of the distance measure are non unique. Therefore regularization becomes inevitable. For regularization, in this paper we focus on regularization functionals involving gradients, for instance

$$\mathcal{R}(u) = \int_\Omega |\nabla u|^p \, dx, \quad p \in \mathbb{N} \,,$$

where $|\nabla u|^p := \sum_{i,j=1}^d \left|\partial_i u^j\right|^p$. Another choice for regularization functional (elastic potential) is discussed in Remark 3. The considered regularized registration problem then read as follows

$$\text{minimize} \quad \mathcal{T}(u) := \mathcal{S}(u) + \alpha \mathcal{R}(u) \quad \text{subject to} \quad u \in \mathcal{A} \,, \tag{2}$$

where $\mathcal{A}$ is an intersection of two of the following sets of constraints:

$$\mathcal{A}_b^p := \{u \in W^{1,p} \mid \|u\|_{L^p} \le b\}, \quad b < \infty\} \,, \tag{3}$$

$$\mathcal{A}_E^{s,p} := \left\{u \in W^{s,p} \mid C(u) = 1 \text{ a.e. in } \Omega\right\} \,, \tag{4}$$

$$\mathcal{A}_I^{s,p} := \left\{u \in W^{s,p} \mid k \le C(u) \le K \text{ a.e. in } \Omega\right\} \,. \tag{5}$$

The set of bound constraints $\mathcal{A}_b^p$ is very general. In particular bounding the displacement by twice the diameter of Ω does not provide a constraint to the registration problem at all. The set $\mathcal{A}_E^{s,p}$ (E stands for equality constraints) is the set of all *volume preserving* transformations. The elements of the sets $\mathcal{A}_I^{s,p}$ allow for some tolerance for local volume preservation. Note that for the particular choices $K(x) \equiv k(x) \equiv 1$ the equality constraints are a special case of the inequality constraints.

Remark 1. For spatial dimension $d = 1$ and a smooth and differentiable displacement u, the condition $u \in \mathcal{A}_E^{1,p}$ implies $u'(x) = 0$ for all $x \in \Omega$. Hence the only feasible transformation is a translation $y(x) = x + b$ with $b \in \mathbb{R}$. For $d = 2$, the situation is already much more complex. For example, any transformation $y(x) = x + u(x)^T$ with $u(x) = (0, g(x_1))^T$ does fulfill the constraints $C(u) = 1$, independent on the choice of g:

$$C(u) = \begin{vmatrix} 1 & 0 \\ g' & 1 \end{vmatrix} = 1 \,.$$

In general, $C(u) = 1$ leads to non-linear differential constraints since the determinant results in a polynomial of degree d in the partial derivatives of u. For example, $\det(\nabla u) = u^1_{x_1} u^2_{x_2} - u^1_{x_2} u^2_{x_1}$ for $d = 2$.

3. **The variational setup.** The goal of this paper is to characterize and identify feasible choices and combinations of regularization functionals and set of constraints guaranteeing the existence of minimizing elements of problem (2). The main result in this section is Theorem 4.1. The first part of this section concentrates on existence results in general and the second part is dedicated to the set of volume constrained functions. Theorem 3.3 states under which assumptions the constraint sets are weakly closed.

To formulate the results of this subsection we use the concept of Carathéodory functions [7, page 74].

Definition 3.1 (Carathéodory [7])**.** Let $\Omega \subset \mathbb{R}^d$ be an open set and let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \overline{R} := \mathbb{R} \cup \{\pm\infty\}$. Then f is a Carathéodory function if

1. $f(x,\cdot,\cdot)$ is continuous for almost every $x \in \Omega$,
2. $f(\cdot,\xi,A)$ is measurable in x for every $(\xi,A) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}$.

It is convenient to rewrite the functional $\mathcal{T}$ from (2) as

$$\mathcal{T}(u) = \int_\Omega f(x,u(x),\nabla u(x))\,dx, \tag{6}$$

with $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \overline{\mathbb{R}}, \quad (x,u,A) \mapsto f(x,u,A)$. For example, choosing $\mathcal{R}(u) = \int_\Omega |\nabla u(x)|^p\,dx$ results in $f(x,\xi,A) = |I_1(x+\xi) - I_0(x)|^2 + \alpha\,|A|^p$.

Finally, if second order derivatives are involved, we use the notation

$$\overline{\mathcal{T}}(u) = \int \overline{f}(x,u(x),\nabla u(x),Hu(x))\,dx\,,$$

where the last component Hu stands for second order terms. Definition 3.1 extends straightforwardly to the higher order case.

3.1. **Existence theorems in the calculus of variations.** We summarize conditions on f which ensure that the constrained problem (2) has a minimizing element. The following results from [7, Chapter 3] are reviewed. For our registration problem we have to modify these results a little bit, since boundary-settings in the original problem are too strong for our application (see Remark 2 for the difference).

Theorem 3.2. *Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \overline{\mathbb{R}}$ be a* Carathéodory *function satisfying the coercivity condition*

$$f(x,u,A) \geq \beta|A|^p + \gamma(x) \tag{7}$$

for almost every $x \in \Omega$, for every $(u,A) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}$, for some function $\gamma \in L^1(\Omega,\mathbb{R})$, $\beta > 0$ and $p > 1$. Assume that f is convex in A. For $u_0 \in W^{1,p}$, let $\tilde{\mathcal{A}} \subset u_0 + W_0^{1,p}$ be a weakly closed set of admissible functions and $\mathcal{T}$ as in (6) defined on $W^{1,p}$. Moreover, assume that there exists $z \in \tilde{\mathcal{A}}$ with $\mathcal{T}(z) < \infty$, then $\mathcal{T}(u)$ attains a minimum.

Proof. The central assumption is that the Carathéodory function f is convex with respect to A. This is a sufficient condition for $\mathcal{T}$ to be weakly lower semi-continuous in $W^{1,p}$ [7, Section 3.3]. From this and the coercivity of f it follows that attains a minimum on $u_0 + W_0^{1,p}$. □

Remark 2. A central assumption of Theorem 3.2 it is that the weakly closed set $\tilde{\mathcal{A}}$ is a subset of $u_0 + W_0^{1,p}$. This means, that for $u \in \tilde{\mathcal{A}}$, $u = u_0$ on $\partial\Omega$, hence in our numerical implementation, we would have to set boundary conditions. Since for our registration problem we do not know u_0 and we do not want to fix u on the boundary of Ω, hence we need to replace the condition $\tilde{\mathcal{A}} \subset u_0 + W_0^{1,p}$ my a more reasonable condition, under which we still can proof existence of a minimizing element (see blow).

In the original proof of Theorem 3.2 in [7, Theorem 4.1, p. 82] the condition $\tilde{\mathcal{A}} \subset u_0 + W_0^{1,p}$ guarantees that a minimizing sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in the L^p-norm (via the Poincaré-inequality). However, in our application all elements of $\mathcal{A}$ are bounded in the L^p-norm, hence it is sufficient to assume that the set of admissible functions is a subset of $W^{1,p}$ instead of $u_0 + W_0^{1,p}$. This implies that no additional boundary conditions on the transformation have to be imposed.

To see that $\mathcal{A}_b^p$ is closed weakly closed in $W^{1,p}$, consider $\mathcal{A}_b^p$ for some $b < \infty$. Note that $\mathcal{A}_b^p$ is convex and closed in $W^{1,p}$. The mapping $M : W^{1,p} \to \mathbb{R}\,,\ u \to \|u\|_{L^p}$ is continuous. Thus $\mathcal{A}_b^{1,p}$ is the pre-image of $[0, b]$ under a continuous mapping, thus it is closed with respect to the $W^{1,p}$-norm. Thus, since $\mathcal{A}_b^p$ is convex and closed in $W^{1,p}$, it is weakly closed in $W^{1,p}$.

The previous theorem can be extended to higher order derivatives.

3.2. **Weak closedness of the sets of constraints.** In the previous subsection, we showed existence of a minimizing element of the registration functional assuming that the sets of constraints $\mathcal{A}$ are closed with respect to the weak topologies (see Theorem 3.2 and Remark 2). We now prove the weak closedness of the sets $\mathcal{A}_E^{s,p}, \mathcal{A}_I^{s,p}$ as defined in (4) and (5), respectively.

Theorem 3.3.

1. *For $d \geq 2$ and $p \geq d$, the sets $\mathcal{A}_I^{1,p}, \mathcal{A}_E^{1,p}$ are weakly closed with respect to the $W^{1,p}$-topology.*
2. *The sets $\mathcal{A}_I^{2,p}, \mathcal{A}_E^{2,p}$ are weakly closed with respect to the $W^{2,p}$-topology.*

Proof. For part (1) we distinguish the cases $p = d$ and $p > d$. For $p > d$, the mapping $M : \mathcal{A}_I^{1,p} \to L^{\frac{p}{d}}(\Omega, \mathbb{R}),\ u \mapsto C(u)$, is continuous with respect to the weak topology on both $W^{1,p}$ and $L^{\frac{p}{d}}(\Omega, \mathbb{R})$; see [11, Section 8.2.4, Lemma, p. 454]. Hence, the set $\mathcal{A}_I^{1,p}$ is a pre-image of the closed set $\{u \in L^{p/d} : k \leq C(u) \leq K \text{ a.e. in } \Omega\}$ under the weakly continuous mapping M with respect to the weak topology on $W^{1,p}$. Thus $\mathcal{A}_I^{1,p}$ is weakly closed.

For $p = d$, we assume $0 < K \leq B < \infty$ and define the set

$$\mathcal{B}^{1,p} := \{u \in W^{1,p} \mid \|C(u)\|_{L^\infty} \leq B\}.$$

First we prove that $\mathcal{B}^{1,p}$ is weakly closed with respect to the $W^{1,p}$-topology, then we show that the mapping $M : \mathcal{B}^{1,p} \to L^q(\Omega, \mathbb{R}),\ u \mapsto C(u)$ is continuous with respect to the weak topology on $W^{1,p}$ and the weak topology on L^q. With this we can argue as before, that $\mathcal{A}_I^{1,p}$ is the pre-image closed set of a weakly closed mapping and consequently it is weakly closed.

Every weak convergent sequence $(u_k)_k$ in $\mathcal{B}^{1,p}$ with weak limit u induces a sequence $c_k := C(u_k)$ in L^∞. Since $\sup_{k\in\mathbb{N}} \{\|c_k\|_{L^\infty}\} \leq B$ and according to the Alaoglu-Bourbaki-Kakutani Theorem, $(c_k)_k$ contains a weak $*$ convergent subsequence $(c_{k_i})_i$ with a weak limit $z \in L^\infty$,

$$\forall\ \phi \in L^1(\Omega, \mathbb{R}) : \quad \lim_{k\to\infty} \int_\Omega C(u_{k_i})\phi \, dx = \int_\Omega z\phi \, dx \tag{8}$$

¿From [7, Chapter 4, Theorem 2.6, p. 172] we know that for $u_k \rightharpoonup_{W^{1,p}} u$, $C(u_k) \rightharpoonup C(u)$ weakly in $\mathcal{D}'(\Omega)$, meaning that

$$\forall\ \phi \in \mathcal{C}_0^\infty : \quad \int_\Omega C(u_k)\phi\ dx \to \int_\Omega C(u)\phi\ dx.$$

Moreover, since $\mathcal{C}_0^\infty(\Omega) \subset L^1(\Omega)$ we have $z = C(u)$ and thus $C(u) \in L^\infty(\Omega)$. Hence $\mathcal{B}^{1,p}$ is weakly closed.

Since Ω is bounded, $L^\infty(\Omega) \subset L^q(\Omega)$ and $L^{q'}(\Omega) \subset L^1(\Omega)$, for $1 < q < \infty$, where $\frac{1}{q} + \frac{1}{q'} = 1$. Hence for $u_k \in \mathcal{B}^{1,p}$, $C(u_k) \in L^q$, and thus (8) also holds for all $\phi \in L^{q'}$. This implies that the mapping M is weakly closed with respect to the weak topology on $W^{1,p}$ and the weak topology on L^q. Thus $\mathcal{A}_I^{1,p} \subset \mathcal{B}^{1,p}$ is the pre-image of the closed set

$$\{u \in L^q(\Omega, \mathbb{R}^d) \mid k \le C(u) \le K \text{ a.e. in } \Omega\}$$

under a weak continuous mapping. Hence, $\mathcal{A}_I^{1,p}$ is weakly closed.

For the second statement we show that $\mathcal{A}_I^{2,p}$ is weakly closed. Using again the result in [7, Chapter 4, Theorem 2.6, p. 172] and the compact imbedding $W^{2,p} \subset W^{1,\frac{dp}{d-p}}$ one can proof that $M : \mathcal{A}_I^{2,p} \to L^{1+\delta}(\Omega, \mathbb{R})$, $u \mapsto C(u)$ is continuous with respect to the weak topologies on $W^{2,p}$ and $L^{1+\delta}(\Omega, \mathbb{R})$. The set $\mathcal{A}_I^{2,p}$ is thus the pre-image of the closed set

$$\{u \in L^{1+\delta} \mid k \le C(u) \le K \text{ a.e. in } \Omega\}$$

under the weak continuous mapping M, and hence weakly closed. □

4. **Minimizing elements for the registration problems.** In the previous section we proved the weak closedness of the constraint sets. We are now ready to prove the existence of minimizing elements for the registration problem (2) with respect to the different constraints. According to Theorems 3.2, it remains to check the following conditions for f (or $\overline{f}$ in case of higher order regularization):

1. f or $\overline{f}$ is a **Carathéodory** function,
2. f or $\overline{f}$ satisfies the **coercivity** condition (7) or $\overline{f}(x, u, A, H) \ge \beta\,|H|^p$, respectively,
3. the **admissible set** of functions $\mathcal{A} \subset \mathcal{A}_b^{1,p}$ is **weakly closed** with respect to $W^{1,p}$; $\mathcal{A} \subset u_0 + W_0^{2,p}$ is weakly closed with respect to $W^{2,p}$.

Theorem 4.1. *Assume that I_0 and I_1 are continuous and that the sets $\mathcal{A}_b^{1,p}$ and $\mathcal{A}_I^{s,p}$ are as in (3) and (5), respectively. For the following constrained image registration functionals there exist minimizing elements:*

1. *for $d \ge 2$, $p \ge d$, and*

$$\mathcal{T}(u) = \mathcal{S}(u) + \alpha \int |\nabla u|^p\, dx \to \min \ \text{ subject to } u \in \mathcal{A}_I^{1,p} \cap \mathcal{A}_b^{1,p},$$

2. *for $d \ge 2$, $p \ge 1$, and higher order regularization*

$$\overline{\mathcal{T}}(u) = \mathcal{S}(u) + \alpha_1 \int_\Omega |\nabla u|^p\, dx + \alpha_2 \int_\Omega |H(u)|^p\, dx \to \min$$

subject to $u \in \mathcal{A}_I^{2,p} \cap \left(u_0 + W_0^{2,p}\right)$. Here $H(u)$ is the hessian of u.

Proof. Since the images I_0 and I_1 are assumed to be continuous, the functions f or $\overline{f}$ are Carathéodory functions.

1. For $d \geq 2$ and $p \geq d$ we have
$$f(x, \xi, A) = (I_1(x+\xi) - I_0(x))^2 + \alpha\, |A|^p\,,$$
thus $\alpha\,|\nabla u|^p \leq f(x, u, \nabla u)$, and hence f is coercive in $W^{1,p}$. Moreover, $\mathcal{A}_b^{1,p} \cap \mathcal{A}_I^{1,p}$ is weakly closed with respect to the $W^{1,p}$-norm; cf. Theorem 3.3. According to Theorem 3.2, $\mathcal{T}$ attains a minimum.
2. For higher order regularization,
$$\overline{f}(x, \xi, A, H) = (I_1(x+\xi) - I_0(x))^2 + \alpha_1 |A|^p + \alpha_2 |H|^p.$$
Since $\alpha_2 |H(u)|^p \leq \overline{f}(x, u, \nabla u, H(u))$ the coercivity condition is satisfied in $W^{2,p}$. The set of admissible functions $\mathcal{A}_I^{2,p} \cap (u_0 + W_0^{2,p})$ is weakly closed with respect to the $W^{2,p}$-norm; cf. Theorem 3.3. An application of a straightforward extension of Theorem 3.2 to higher order derivatives completes the proof.

□

Remark 3 (Elastic Regularization Functional)**.** Theorem 4.1 indicates that in our existence analysis, the spacial dimension d influences the choice of the regularization functional $\mathcal{R}$ strongly. The limiting factor in this existence result is the determinant constraint, since the determinant is a polynomial of degree d. For example, the probably most commonly used *elastic regularization* [1, 3, 2, 5, 14, 17] is given by

$$\mathcal{R}_{\text{elas}}(u) := \sum_{i=1..d} \sum_{j=1..d} \int_\Omega \left(\frac{\lambda_1}{2} \frac{\partial\, u^i}{\partial x_i} \frac{\partial\, u^j}{\partial x_j} + \frac{\lambda_2}{4} \left(\frac{\partial\, u^i}{\partial x_j} + \frac{\partial\, u^j}{\partial x_i} \right)^2 \right) dx\,, \tag{9}$$

with material parameters $\lambda_1, \lambda_2 > 0$. Using $\min\{\lambda_1/2, \lambda_2/4\} \int_\Omega |\nabla u|^2 \leq \mathcal{R}_{\text{elas}}(u)$, and Theorem 4.1, existence of a minimizing element for the volume preserving constraint registration functional is only guaranteed for $d = 2$.

5. Finite dimensional approximation.

In this section we study a finite dimensional approximation of the minimization problem in (2). The main result of this section is stated in Section 5.3, where we give necessary assumptions under which the solutions of the discretized minimization problem converge to a solution of the original registration problem.

An index m is connected to the approximation of the images, an index n to the approximation of the Sobolev spaces, and $\mathcal{A}_n$ are the approximated spaces of functions, satisfying the constraints as explained in the next section.

5.1. Approximation of the minimization functional.

We assume that $I_0, I_1 \in \mathcal{C}_0^0(\mathbb{R}^d, \mathbb{R})$ and that the components of ∇I_1 are bounded by $C_{I_1} := \|\nabla I_1\|_{L^\infty}$. We consider the following operator

$$F : W^{1,2} \to L^2(\mathbb{R}^d, \mathbb{R}), \quad u \mapsto I_1(id + u). \tag{10}$$

We approximate the images I_l by piecewise affine functions such the approximated image $I_{l,m}$, satisfies $\|I_{l,m} - I_l\|_{L^2}^2 \leq \delta_m^l$. The parameter m is connected to the image-resolution. Moreover we define the approximated operator F_m by

$$F_m : W^{1,2} \to L^2(\mathbb{R}^2, \mathbb{R}), \quad u \mapsto I_{1,m} \circ (id + u). \tag{11}$$

Since we have assumed that the images I_l are continuous, the operator F in (10) is compact and satisfies a Lipschitz condition; cf. Lemma 5.1. This properties are exploited in Theorem 5.5 which states convergence of the solutions of the approximated problem to a solution of the original problem.

In the following we consider the case of inequality constraints, i.e, the sets $\mathcal{A}_I^{s,p}$.

Lemma 5.1. *Assume that* $I_1 \in \mathcal{C}_0^0(\mathbb{R}^d, \mathbb{R})$, *and* F, F_m *as in* (10),(11). *Then*

1. *F is compact,*
2. *F is Lipschitz continuous: there exists $C_F > 0$ such that $\|F(v) - F(u)\| \leq C_F \|v - u\|$,*
3. *Assume that $0 < k \leq \det(id + u) \leq K$, then*

$$\|F_m(u) - F(u)\|^2 \leq k^{-1}\delta_m^1 \quad \textit{for all } u \in \mathcal{D}(F) \cap \mathcal{A}_I^{1,d} \, . \tag{12}$$

Proof.

1. First we prove that F is compact. Assume therefore that we have a $\|\cdot\|_{W^{1,2}}$-bounded sequence $\{u_i\}$, which defines a sequence $\{F^i\}$ in $L^2(\mathbb{R}^d, \mathbb{R})$, by the relation $F^i := F(id + u_i) = I_1 \circ (id + u_i)$. Since $I_1 \in \mathcal{C}_0^0(\mathbb{R}^2, \mathbb{R})$, F^i is bounded in $W^{1,2}$, which can be seen by the following inequalities
$$\begin{aligned}\left\|F^i\right\|_{L^2} &\leq \|I_1(id + u_i) - I_1(id) + I_1(id)\|_{L^2}\\ &\leq \|I_1(id + u_i) - I_1(id)\|_{L^2} + \|I_1(id)\|_{L^2}\\ &\leq C_{I_1}\|u_i\|_{L^2} + \|I_1(id)\|_{L^2}\end{aligned}$$
and $\left\|\nabla F^i\right\|_{L^2} \leq \|\nabla I_1(id + u_i)\nabla(id + u_i)\|_{L^2} \leq C_{I_1}\|\nabla(id + u_i)\|_{L^2}$. Thus according to the assumptions $\left\{F^i\right\}$ is bounded and has a weakly convergent subsequence $\left\{F^{i_k}\right\}$ in $W^{1,2}$. Using Sobolev embeddings it follows that $\left\{F^{i_k}\right\}$ is strongly convergent in the L^2-norm. Hence F is compact.
2. For arbitrary $x, \tilde{x} \in \Omega$ we have $|I_1(x) - I_1(\tilde{x})| \leq C_{I_1}|x - \tilde{x}|$. Thus we get the following estimate:
$$\|F(v) - F(u)\|_{L^2} = \|I_1 \circ (id + v) - I_1 \circ (id + u)\|_{L^2} \leq C_{I_1}\|v - u\|_{L^2} \ .$$
The Lipschitz-constant of F is less or equal to C_{I_1}.
3. Since we assume that $0 < k \leq \det(id + u) \leq K$ we can use the transformation formula and obtain
$$\begin{aligned}&k \int_\Omega |I_1(x + u(x)) - I_{1,m}(x + u(x))|^2\, dx\\ &\leq \int_\Omega |I_1(x + u(x)) - I_{1,m}(x + u(x))|^2 \det(\nabla(id + u)(x))dx\\ &= \int_\Omega |I_1(x) - I_{1,m}(x)|^2\, dx \leq \delta_m^1 \ .\end{aligned}$$

□

In the following we consider the approximated functionals. Setting $\mathcal{S}_m(u) := \|I_{1,m} \circ (id + u) - I_{0,m}\|^2$ and $\mathcal{R}(u) := \|\nabla u\|_{L^d}^d$, the approximated objective functional reads

$$\mathcal{T}_{m,n}(u) = \mathcal{S}_m(u) + \alpha\mathcal{R}(u) \longrightarrow \min \ \text{over } \mathcal{A}_n. \tag{13}$$

5.2. **Approximation of the constraints.** The final step is the approximation of the box constraints $\mathcal{A}_I^{1,d}$ by a sequence of finite dimensional subspaces $\mathcal{A}_n$. We aim for an approximation which allows easy handling of the determinant-constraint. In particular, we choose linear finite elements on triangles or tetrahedrons of polynomial degree one. With these the determinant of the approximation is a constant on each triangle of the triangulation.

In the two dimensional case, starting with a triangulation as shown in Figure 1, a refinement is obtained by dividing each triangle into four congruent triangles. This leads to a family of regular triangulations $\Gamma_n := (\tau_1, \cdots, \tau_{2\cdot 2^{2n}})$. Analogously we handle the 3d-triangulation, which is denoted again by Γ_n.

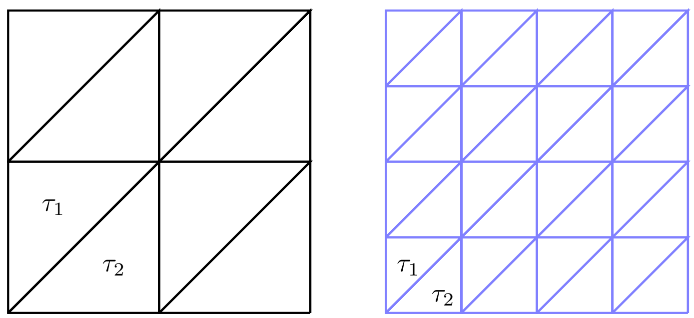

FIGURE 1. Refinement of the triangulation. Left: Γ_1, right: Γ_2.

The displacements are elements of the following set:

$$\mathcal{U}^n := \left\{ u \in \mathcal{C}^0(\Omega, \mathbb{R}^d) \mid u_{|\tau_i} \in \Pi_1(\Omega, \mathbb{R}^d) \text{ for every } \tau_i \in \Gamma_n \right\},$$

where Π_1 is the set of polynomials of degree 1. By this choice of the refinement we have a nested sequence of spaces

$$\cdots \subset \mathcal{U}^n \subset \mathcal{U}^{n+1} \subset \cdots \bigcup_{m \in \mathbb{N}} \mathcal{U}^m \quad \overline{\bigcup_{n \in \mathbb{N}} \mathcal{U}^n} = W^{2,2}.$$

For $d = 2$, $u_{|\tau_i}$ is of the form:

$$u_{|\tau_i}(x) = \begin{pmatrix} a_i^1 + b_i^1 x_1 + c_i^1 x_2 \\ a_i^2 + b_i^2 x_1 + c_i^2 x_2 \end{pmatrix} \quad \text{and hence} \quad \nabla u_{|\tau_i}(x) = \begin{pmatrix} b_i^1 & c_i^1 \\ b_i^2 & c_i^2 \end{pmatrix}.$$

The challenging part in the registration problem is to incorporate the determinant constraints. For the ease of presentation, we restrict ourself to the case of constant bounds k and K (that appear in definition of $\mathcal{A}_I^{s,p}$). The case of non-constant box constraints is along the same lines.

We distinguish two cases: the **integrated or global** and the **local** constraints.

- The integrated constraints are based on the L^1-norm, and give the constraint a global nature. The bound for the discrepancy is a function of the discretization parameter h_n, i.e. basically the size of the triangles/tetrahedrons. Note that $\|\det(Id + \nabla u_n) - 1\|_{L^1} \le \epsilon(h_n)$ does not prevent $\det(Id + \nabla u_n)$ to be negative in some of the triangles. Here $Id \in \mathbb{R}^{d \times d}$ denotes the identity matrix.

  For the global L_1-norm based integrated determinant-constraint we work with

$$\mathcal{A}^1_{\epsilon(h_n)} := \{ u_n \in \mathcal{U}^n \mid \|\det(Id + \nabla u_n) - 1\|_{L^1} \le \epsilon(h_n) \}. \tag{14}$$

In the $2d$ case this reads

$$\mathcal{A}^1_{\epsilon(h_n)} = \left\{ u_n \in \mathcal{U}^n \mid \frac{h_n^2}{2} \sum \left|(1+b_i^1)(1+c_i^2) - c_i^1 b_i^2 - 1\right| \leq \epsilon(h_n) \right\}.$$

The reason for this choice is that under the right choice of the tolerance $\epsilon(h_n)$ in dependence of h_n, (see Lemma 5.3) we can use the $W^{1,2}$-norm projection

$$P_n^1 : W^{1,2} \to \mathcal{U}^n, \quad u \mapsto \operatorname{argmin}_{u_n \in \mathcal{U}^n} \|u - u_n\|^2_{W^{1,2}}. \tag{15}$$

in order to project onto this space. As a consequence we have to specify a condition on $\epsilon(\cdot)$ that guarantees that $P_n^1(u) \in \mathcal{A}^1_{\epsilon(h_n)}$. This condition is given in Lemma 5.3.

- The measure for the local determinant-constraints is based on the L^∞-norm. This implies that we take the maximal change of the determinant on each triangle. A disadvantage of this choice is that we cannot guarantee that $W^{1,2}$-norm projected functions are elements of the set of functions satisfying the local determinant constraints. Hence we have to introduce an alternative projection operator. Since the corresponding sets are not convex, the projection onto them does not have to be unique. For the local L^∞-norm based determinate-constraints we work with

$$\mathcal{A}^\infty_{n,\bar{\epsilon}} := \{u_n \in \mathcal{U}^n \mid \|\det(\nabla u_n + I)\|_{L^\infty} \leq \bar{\epsilon}\}, \tag{16}$$

with a constant $\bar{\epsilon}$. For the $2d$ case this reads:

$$\mathcal{A}^\infty_{n,\bar{\epsilon}} = \left\{ u_n \in \mathcal{U}^n \mid \left|(b_i^1+1)(c_i^2+1) - c_i^1 b_i^2 - 1\right| \leq \bar{\epsilon}, i = 1 \ldots 2 \cdot 2^{2n} \right\}.$$

As projection operator we choose

$$P_n^\infty : W^{1,2} \to \mathcal{A}^\infty_{n,\bar{\epsilon}}, \quad u \mapsto \operatorname{argmin}_{u_n \in \mathcal{A}^\infty_{n,\bar{\epsilon}}} \|u - u_n\|^2_{W^{1,2}}. \tag{17}$$

When minimizing over $\mathcal{A}^\infty_{n,\bar{\epsilon}}$, we solve a finite dimensional minimization problem. Existence of a minimum is assured, due to the continuity of the absolute value and the determinant, in the finite dimensional setting. In our algorithm, the condition $u_n \in \mathcal{A}^\infty_{n,\bar{\epsilon}}$ is realized via a Lagrangian method [28, p. 317–319], described in Section 6.

In the following we provide a Lemma that gives a condition on the function $\epsilon(h_n)$ used for the definition of the global determinant constraint sets, that assures that the least squares approximation $P_n^1(u)$ in (15) is an element of the set of integrated determinant constraints. A central estimation in the proof of Lemma 5.3 is given by the following theorem, see Cialet [6, Theorem 18.1. , p 138] or [10, Corollary 110, p. 61–62].

Theorem 5.2 (Approximation of $W^{2,2}$ functions)**.** *There exists a constant $C_\Omega < \infty$ such that*

$$\inf_{u_n \in \mathcal{U}^n} \|u - u_n\|_{W^{1,2}} \leq C_\Omega \|u\|_{W^{2,2}} h_n$$

for every $u \in W^{2,2}$.

The following Lemma suggests the choice of $\epsilon(h)$ as in (18).

Lemma 5.3. *Let $u \in W^{2,2}$ and h_n be the mesh size parameter. If*

$$\epsilon(h_n) \geq C_\Omega C_{det} \|id + u\|_{W^{2,2}} h_n, \tag{18}$$

then $P_n^1(u) \in \mathcal{A}^1_{\epsilon(h_n)}$. The constant C_Ω depends on Ω and the regularity of the triangulation. The constant C_{det} depends on the space dimension d and on ∇u.

Proof. Given $A \in \mathbb{R}^{d\times d}$, for $1 \leq i, j \leq d$, define $A_S^{ij} \in \mathbb{R}^{d-1\times d-1}$ to be the sub matrix of A, formed by removing from A its i^{th} row and j^{th} column. Then we can write

$$\det(A) = \sum_{j=1}^{d} (-1)^{i+j} A_{ij} \det(A_S^{ij}) \ .$$

For the gradient of the the determinant-function we obtain

$$\nabla \det(A) = \Big((-1)^{i+j} \det(A_S^{ij}) \Big)_{1\leq i,j\leq d} \ .$$

We denote $|M|_{\mathbb{R}^{d\times d}} := \sum_{1\leq i,j\leq d} |M_{ij}|$, and $\langle M, M\rangle_{\mathbb{R}^{d\times d}}$ is the point wise product of matrices. Let $A_n \in \mathbb{R}^{d\times d}$, then Taylor-expansion gives:

$$\det(A) = \det(A_n) - \langle \nabla \det(A_n), A - A_n\rangle_{\mathbb{R}^{d\times d}} + O((A-A_n)^2) \ . \tag{19}$$

The entries of $|\nabla \det(A)|$ are polynomials of degree $d-1$ in the variables A_{ij}. With this we can estimate the norm by

$$\begin{aligned} |\nabla \det(A)|_{\mathbb{R}^{d\times d}} &\leq |A|_{\mathbb{R}^{2\times 2}} := c(A,2), && \text{for } d=2 \\ |\nabla \det(A)|_{\mathbb{R}^{d\times d}} &\leq 2\,|A|^2_{\mathbb{R}^{3\times 3}} := c(A,3) && \text{for } d=3 \ . \end{aligned}$$

which together with (19) implies

$$\begin{aligned} |\det(A_n) - \det(A)| &= |\langle \nabla \det(A), A - A_n\rangle_{\mathbb{R}^4}| + O\left((A-A_n)^2\right) \\ &\leq |\nabla \det(A)|\,|A-A_n| + O\left((A-A_n)^2\right) \\ &\leq c(A,d)\,|A-A_n| + O\left((A-A_n)^2\right) \ . \end{aligned}$$

For $v = id + u \in W^{2,2}$ Theorem 5.2 states that $\left\|P_n^1(v) - v\right\|_{W^{1,2}} \leq C_\Omega \left\|v\right\|_{W^{2,2}} h_n$. Set

$$C_{det} := \begin{cases} \|c(\nabla u,3)\|^2_{L^2} = \int_\Omega |\nabla u(x)|^2\,dx & \text{for } d=2 \\ \|c(\nabla u,3)\|^2_{L^2} = \int_\Omega 2\,|\nabla u(x)|^4\,dx & \text{for } d=3. \end{cases}$$

Then we obtain following estimate:

$$\begin{aligned} &\left\|\det\left(\nabla P_n^1(v)\right) - \det\left(\nabla v\right)\right\|_{L^1} \\ &= \int_\Omega \left|\det\left(\nabla P_n^1(v)(x)\right) - \det\left(\nabla v(x)\right)\right| dx \\ &\leq \int_\Omega c(\nabla v(x),d)\left|\nabla P_n^1(v)(x) - \nabla v(x)\right| dx + O\left(\left\|\nabla P_n^1(v) - \nabla v\right\|^2_{L^2}\right) \\ &\leq \|c(\nabla v,d)\|_{L^2}\left\|\nabla P_n^1(v) - \nabla v\right\|_{L^2} + O\left(\left\|\nabla P_n^1(v) - \nabla v\right\|^2_{L^2}\right) \\ &\leq C_{det}\, C_\Omega\, \|v\|_{W^{2,2}}\, h_n \ . \end{aligned}$$

Thus if $\epsilon(h_n) \geq C\,\|v\|_{W^{2,2}}\,\|\nabla v\|_{L^2}\,h_n$ then $P_n^1(v) \in \mathcal{A}^1_{\epsilon(h_n)}$ (set with integrated determinant constraints). □

A necessary ingredient for the convergence of solutions of the discretized problems to a solution of the inverse problem is that the projection operators converge to the identity.

Theorem 5.4. *Let $\mathcal{A}^1_{\epsilon(h_n)}, \mathcal{A}^\infty_{n,\bar{\epsilon}}$ be as in (15), (16) and P_n^1, P_n^∞ as in (15), (17).*

1. Equality constraints: *Assume that* $u \in \mathcal{A}_E^{2,d}$, $\lim_{n\to\infty} \epsilon(h_n) = 0$ *and*
$$\epsilon(h_n) \geq C \left\| \nabla u \right\|_{L^2} \left\| u \right\|_{W^{2,2}} h_n \,.$$
For P_n^1 *as in* (15), $\lim_{n\to\infty} \left\| P_n^1(u) - u \right\|_{W^{1,2}} = 0$.
2. Box constraints: *Assume* $u \in \mathcal{A}_I^{2,d}$ *with constant bounds* $k = 1-\bar{\epsilon}, K = 1+\bar{\epsilon}$. *For* P_n^∞ *as in* (17) *we have* $\lim_{n\to\infty} \left\| P_n^\infty(u) - u \right\|_{W^{1,2}} = 0$.

Proof. For the first part we use that P_n^1 is the $W^{1,2}$-least square spline approximation of u onto $\mathcal{U}^n$ and $u \in \mathcal{A}_E^{2,d} \subset W^{2,d} \subset W^{2,2}$. Hence we can apply Theorem 5.2. Moreover Lemma 5.3 states that $P_n^1(u) \in \mathcal{A}^1_{\epsilon(h_n)}$ by the choice of $\epsilon(h_n)$.

For the second part, we denote with $\bar{\ }$ the closure in $W^{1,2}$. Note that by the choice of $k = 1-\bar{\epsilon}, K = 1+\bar{\epsilon}$, we have that $\mathcal{A}^\infty_{n,\bar{\epsilon}} = \mathcal{U}^n \cap \mathcal{A}_I^{1,d}$. Since $\mathcal{U}^n$ is dense in $W^{2,2}$ [6], we have
$$\overline{\bigcup_{n\in\mathbb{N}} \mathcal{A}^\infty_{n,\bar{\epsilon}}} = \overline{\bigcup_{n\in\mathbb{N}} \mathcal{U}^n \cap \mathcal{A}_I^{1,2}} = \overline{\bigcup_{n\in\mathbb{N}} \mathcal{U}^n} \cap \mathcal{A}_I^{1,2} = W^{2,2} \cap \mathcal{A}_I^{1,2} = \mathcal{A}_I^{2,2} \,.$$
Thus, for $u \in \mathcal{A}_I^{2,d}$ it holds that $\lim_{n\to\infty} \left\| u - P_n^\infty(u) \right\|_{W^{1,2}} = 0$. □

5.3. **Convergence of the approximate solutions.** The proof of the theorem that finite dimensional solutions converge to a solution of the registration problem is based on the following assumptions and definitions.

1. *Constraints:* Let $\mathcal{A}_{con}$ denote the constraints and satisfying the volume preserving constraints, i.e $\mathcal{A}_{con} = \mathcal{A}_E^{1,d} \cap \mathcal{A}_b^d$ or $\mathcal{A}_{con} = \mathcal{A}_I^{1,d} \cap \mathcal{A}_b^d$ (see (3)-(5) for the definition of the sets). Assume $u \in \mathcal{A}_{con} \cap W^{2,d}$. Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ denote either $\mathcal{A}^1_{\epsilon(h_n)}$ with operator P_n^1 or $\mathcal{A}^\infty_{n,\bar{\epsilon}}$ with operator P_n^∞.
2. Define $\mathcal{D}^n := \mathcal{D}(F) \cap \mathcal{A}_n$, where $\mathcal{D}(F)$ is the domain of F (see (10)). Since $0 \in \mathcal{D}^n$, it follows that $\mathcal{D}^n \neq \emptyset$.
3. *Images:* For $l = 0, 1$, let $I_l \in \mathcal{C}_0^0(\mathbb{R}^2, \mathbb{R})$, with approximations $I_{l,m}$ as outlined in Section 5.1; thus, $\left\| I_l - I_{l,m} \right\|_{L^2}^2 \leq \delta_m^{I_l} \to 0$ as $m \to \infty$; set $\delta_m := k^{-1}\delta_m^{I_1} + \delta_m^{I_0}$, where k is the lower bound for the determinant constraint, as in Lemma 5.1.
4. let $\alpha = \alpha(m, n)$ such that for $m, n \to \infty$ it holds that

$$\alpha \to 0, \quad \delta_m/\alpha \to 0, \quad \left\| u^\dagger - P_n(u^\dagger) \right\|_{W^{1,2}}^2 /\alpha \to 0. \tag{20}$$

The following result can be found in a slightly modified version in [26] and is the main result of this section. However, in [26] they consider only the special case with a regularization function $\mathcal{R}(u) = \left\| u - u_0 \right\|_H^2$ and projection $P_n : H \to H$, where H is a Hilbert space.

Theorem 5.5. *Let* $u = u^\dagger$ *be a solution of the inverse problem* $F(u) = I_0$*. Under the above assumptions and with* $\mathcal{R}(u) := \left\| \nabla u \right\|_{L^d}^d$*, the sequence* $\{u_{m_k,n_k}(\alpha_k, \delta_k)\}$ *has a convergent subsequence. The limit of every convergent subsequence is an* $\mathcal{R}$*-minimizing solution.*

If in addition the $\mathcal{R}$*-minimizing solution* $u^\dagger$ *is unique, then*
$$\lim_{\delta_m\to 0, m\to\infty, n\to\infty} u_{m,n} = u^\dagger \,.$$

Proof. Let n be large enough, then according to Theorem 5.4, $P_n(u^\dagger) \in \mathcal{A}_n$. By definition $F(u^\dagger) = I_0$ and by assumption we have $\left\| I_0 - I_{0,m} \right\|^2 \leq \delta_m^{I_0}$. Moreover according to Lemma 5.1 the conditions on $I_{1,m}$ imply that $\left\| F_m(P_n(u^\dagger)) - F(P_n(u^\dagger)) \right\|^2 \leq$

$C\delta_m$. Since $u_{m,n}$ minimizes (13) we have

(21)

$$
\begin{aligned}
&\|F_m(u_{m,n}) - I_{0,m}\|^2 + \alpha\mathcal{R}(u_{m,n}) \leq \\
&\qquad\leq \left\|F_m(P_n(u^\dagger)) - I_{0,m}\right\|^2 + \alpha\mathcal{R}(P_n(u^\dagger)) \\
&\qquad\leq 2\left[\left\|F_m(P_n(u^\dagger)) - F(P_n(u^\dagger))\right\|^2 + \left\|F(P_n(u^\dagger)) - F(u^\dagger)\right\|^2 + \right. \\
&\qquad\qquad \left.\left\|F(u^\dagger) - I_{0,m}\right\|^2\right] + \alpha\mathcal{R}(P_n(u^\dagger)) \\
&\qquad\leq 2\left(k^{-1}\delta_m^{I_1} + C_F^2\left\|u^\dagger - P_n(u^\dagger)\right\|^2 + \delta_m^R\right) + \alpha\mathcal{R}\left(P_n(u^\dagger)\right) .
\end{aligned}
$$

¿From this we obtain

$$
\begin{aligned}
\|F_m(u_{m,n}) - I_{0,m}\|^2 \leq &\left(k^{-1}\delta_m^{I_1} + C_F^2\left\|u^\dagger - P_n(u^\dagger)\right\|^2 + \delta_m^R\right) + \\
&\alpha\left|\mathcal{R}(P_n(u^\dagger)) - \mathcal{R}(u_{m,n}))\right| .
\end{aligned}
$$

Taking the limit $m, n \to \infty$, we know from the assumptions on α, δ_m and P_n that $\left\|u^\dagger - P_n(u^\dagger)\right\|_{L^2} \to 0$ and $\alpha \to 0$. Hence

$$
\|F_m(u_{m,n}) - I_{0,m}\| \to 0 . \tag{22}
$$

Moreover, from (21) together with Theorem 5.4 implying that $\mathcal{R}\left(P_n(u^\dagger)\right) \to \mathcal{R}(u^\dagger)$ and the assumptions on α it follows that

$$
\begin{aligned}
&\liminf \mathcal{R}(u_{m,n}) \\
&\quad\leq \quad \liminf \alpha^{-1}\left(k^{-1}\delta_m^{I_1} + C_F^2\left\|u^\dagger - P_n(u^\dagger)\right\|^2 + \delta_m^R\right) + \mathcal{R}\left(P_n(u^\dagger)\right) \\
&\quad\leq \quad \limsup \alpha^{-1}\left(k^{-1}\delta_m^{I_1} + C_F^2\left\|u^\dagger - P_n(u^\dagger)\right\|^2 + \delta_m^R\right) + \limsup \mathcal{R}\left(P_n(u^\dagger)\right) \\
&\quad\underset{(20)}{=} \quad 0 + \limsup \mathcal{R}(P_n(u^\dagger)) = \mathcal{R}(u^\dagger).
\end{aligned}
$$

Since $\{u_{m,n}\}$ is a $W^{1,2}$-bounded sequence, we know that $u_{m,n}$ has a weakly convergent subsequence with limit $\overline{u} \in U$. For the sake of simplicity of notation, we denote this weakly convergent subsequence again with $\{u_{m,n}\}$. Moreover taking into account (21), the fact that $u^\dagger$ is an $\mathcal{R}$-minimizing solution and the weak-lower semi continuity of $\mathcal{R}$, it follows that

$$
\mathcal{R}\left(\overline{u}\right) \leq \liminf \mathcal{R}\left(u_{m,n}\right) \leq \mathcal{R}(u^\dagger) .
$$

Thus, $\mathcal{R}\left(\overline{u}\right) = \lim \mathcal{R}\left(u_{m,n}\right) = \mathcal{R}(u^\dagger)$ and $I_0 = \lim F(u_{m,n}) = F(\overline{u})$.

Moreover, (12) implies that for any subsequence u_{m_k,n_k} of $u_{m,n}$

$$
\begin{aligned}
\|F(u_{m_k,n_k}) - I_0\|^2 &\leq 2\|F(u_{m_k,n_k}) - F_m(u_{m_k,n_k})\|^2 + 2\|F_{m_k}(u_{m_k,n_k}) - I_0\|^2 \\
&\leq \underbrace{2k^{-1}\delta_{m_k}^{I_1}}_{\to 0} + \underbrace{4\|F_{m_k}(u_{m_k,n_k}) - I_{0,m}\|^2}_{\to_{(22)} 0} + \underbrace{4\delta_m^{I_0}}_{\to 0} .
\end{aligned}
$$

Taking the limit $m_k, n_k \to \infty$, this implies that

$$
\|F(u_{m_k,n_k}) - I_0\|^2 \to 0 ,
$$

Hence $\overline{u}$ is an $\mathcal{R}$-minimizing solution. □

6. **Solving the registration problem with finite elements.** To show that this theory can be applied to determinant based constraints, we present a simplified example for the equality case. An implementation of the more general box constraints is subject of current research. In order to solve the registration problem numerically we use a finite element approach. In particular, we use triangles and tetrahedron to ensure a nested sequence of the finite element spaces. Particularly, we use the least squares functional $\mathcal{S}$ in (1) to compare the reference and the deformed template image and $\mathcal{R} = \mathcal{R}_{elast}$ (elastic) as in (9) or $\mathcal{R}(u) = \|\nabla u\|_{L^3}^3$ (cubic) as regularization functionals. We note that for the elastic regularizer the existence of a minimizing element is only guaranteed in the $2d$ case.

We apply the **Augmented Lagrangian method** [28] to incorporate the determinant constraints $\det(\nabla u + Id) = 1$, that is, we iterate the following minimization problem

(23)

$$\begin{aligned} u_k \in \operatorname{argmin}_{u \in U_n} \Big\{ & \underbrace{\mathcal{S}(u) + \alpha \mathcal{R}(u) + \beta \|u - \tilde{u}_l\|_{L^2}^2}_{\mathcal{T}(u)} + ... \\ & .. \frac{\kappa_k}{2} \int_\Omega (\det(\nabla u + Id) - 1)^2 - \int p_{k-1} (\det(\nabla u + Id) - 1) \Big\} \\ & p_k = p_{k-1} + \kappa_k (1 - \det(\nabla u_k + Id)) \end{aligned}$$

in order to minimize $\mathcal{S}(u) + \alpha\mathcal{R}(u)$ under the local determinant equality constraints with $\bar{\epsilon} = 0$ (volume/area preserving).

We do not set any boundary conditions, instead, we achieve that u_k stays bounded by adding an additionally term $\beta \|u - \tilde{u}_l\|^2$, that stabilizes the minimization process (steepest decent iteration) and vanishes, if the minimizing sequence converges to a minimum of (23) with $\beta = 0$. We solve (23) with a semi-implicit gradient decent method. In each iteration step (for the minimization of (23)) we have to update the stiffness-matrix, and $\tilde{u}_l$. We mention again that we do not minimize over $W_0^{1,3}$ but over $W^{1,3}$. Since we bound u, we can still guarantee the existence of a solution (see Remark 2).

For the cubic regularization, in order to avoid solving a nonlinear problem, we approximate the gradient of the cubic regularization functional by

$$\mathcal{R}(u, v) = \frac{1}{3} \sum_{i,j=1..3} \int \left| u_{x_j}^i \right| u_{x_j}^i v_{x_j}^i \sim \sum_{i,j=1..3} \int \left| (\tilde{u}_l)_{x_j}^i \right| u_{x_j}^i v_{x_j}^i$$

where again $\tilde{u}_l$ is the solution of the previous step in the gradient decent minimization process.

6.1. **Numerical example.** The above scheme was implemented in C++ (2D and 3D) using the imaging2 class written by Matthias Fuchs [13]. The imaging2 class provides an object-oriented implementation of basic mathematical objects and functions used in image processing. It includes a FEM module that provides functions to assemble the stiffness matrix and force vector for user-defined equations.

The results for a simple 3D-example are shown in Table 1 and Figures 2 and 4.

7. **Conclusions.** In this paper we have investigated the existence of minimizing elements of area/volume preserving registration functionals. One motivation for

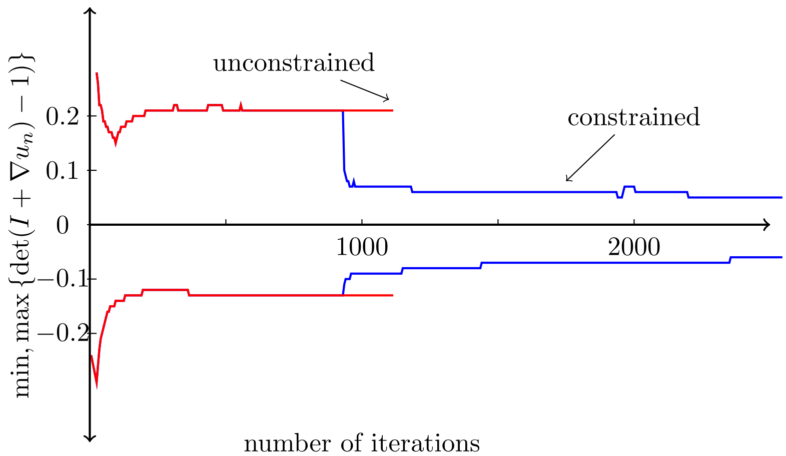

FIGURE 2. Minimal and maximal values of $\det(I+\nabla u_n)-1$. The jumps in the plot for the constraint case, indicate an update of the Lagrange parameter p_k.

	1 (UC)	2	3
error: $\|I_1(u+id)-I_0\|_{L^2}^2$	0.26	0.11	0.22
$\min(\det(id+u)-1)$	-0.13	-0.06	-0.08
$\max(\det(id+u)-1)$	0.21	0.05	0.06

TABLE 1. Parameters of the example shown in Figure 4: $\beta=1$, Example 1 (unconstrained): $\kappa_k=0,\lambda_1=0.1,\lambda_2=0.2$. Example 2 (constrained, elastic regularization): $\kappa_1=0.1,\lambda_1=0.1,\lambda_2=0.2$. Example 3 (constrained, cubic regularization $\mathcal{R}(u)=\|\nabla u\|_{L^2}^3$: $\kappa_1=0.1,\alpha=1$.

studying these is due to our previous work, where we introduced numerical methods for volume preserving image registration [17]. Here we used variational techniques to prove the existence of minimizers of the registration functional. Moreover we provide convergence analysis of the finite dimensional approximation of the minimization problem, clarified the difficulties caused by the discretization of the area/volume preserving constraints and proposed two ways to approximate the set of area constrained transformations.

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REFERENCES

[1] R. Bajcsy and S. Kovačič, *Multiresolution elastic matching*, Computer Vision, Graphics and Image Processing, **46** (1989), 1–21.

[2] M. Bro-Nielsen, "Medical Image Registration and Surgery Simulation," PhD thesis, IMM, Technical University of Denmark, 1996.

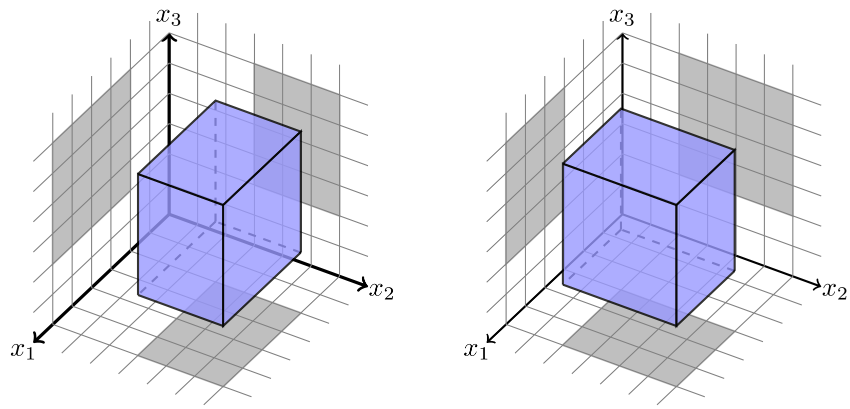

Figure 3. Reference and template image. We shrink the box in one direction and enlarge it in the other one, such that the volumes of the two boxes are equal. The grid squeezes iteratively, such that the volumes of the tetrahedrons remain (almost) equal in each iteration step.

[3] C. Broit, "Optimal Registration of Deformed Images," PhD thesis, Computer and Information Science, University of Pensylvania, 1981.

[4] V. Camion and L. Younes, *Geodesic interpolating splines*, in Proceedings of EMMCVPR 01, M. Figueiredo, J. Zerubia and A. K. Jain, eds., vol. **2134**, LNCS, (2001), 513–527.

[5] G. Christensen, "Deformable Shape Models for Anatomy," PhD thesis, Washington University, Dept. of Electrical Engineering., 1994.

[6] P. Ciarlet and J. Lions, "Handbook of Numerical Analysis," vol. **II**, North-Holland, Amsterdam, Netherlands, 1993. Finite Element Methods (Part 1).

[7] B. Dacorogna, "Direct Methods in the Calculus of Variations," Springer–Verlag, Berlin, 1989.

[8] M. Droske and M. Rumpf, *A variational approach to nonrigid morphological image registration*, SIAM J. Appl. Math., **64** (2003/04), 668–687 (electronic).

[9] P. Dupuis, U. Grenander and M. I. Miller, *Variational problems on flows of diffeomorphisms for image matching*, Quarterly of Applied Mathematics, **56** (1998), 587–600.

[10] A. Ern and J. Guermond, "Theory and Practice of Finite Elements," vol. **159** of Applied Mathematical Sciences, Springer-Verlag, New York, 2004.

[11] L. C. Evans, "Partial Differential Equations," vol. **3** of Graduate Studies in Mathematics, American Mathematical Society, Providence, Rhode Island, third ed., 1998.

[12] J. M. Fitzpatrick, D. L. G. Hill and C. R. J. Maurer, *Image registration*, in Handbook of Medical Imaging, Volume **2**: Medical Image Processing and Analysis, M. Sonka and J. M. Fitzpatrick, eds., SPIE, 2000, 447–513.

[13] M. Fuchs, "Reconstruction of Shapes with A-Priori Knowledge," PhD thesis, Leopold-Franzens-Universität Innsbruck, September 2008.

[14] J. Gee, D. Haynor, L. Le Briquer and R. Bajcsy, *Advances in elastic matching theory and its implementation*, in CVRMed, 1997, 63–72.

[15] E. Haber, S. Heldmann and J. Modersitzki, *A framework for image-based constrained registration with an application to local rigidity*, LAA, (2008), 1–16. Submitted.

[16] E. Haber and J. Modersitzki, *Numerical methods for volume preserving image registration*, Inverse Problems, Institute of Physics Publishing, **20** (2004), 1621–1638.

[17] ———, *Numerical methods for volume preserving image registration*, Inverse Problems, Institute of Physics Publishing, **20** (2004), 1621–1638.

[18] ———, *Image registration with a guaranteed displacement regularity*, IJCV, 1 (2006), pp. 327–330. DOI: 10.1007/s11263-006-8984-4.

[19] S. Haker, S. Angenent, A. Tannenbaum and R. Kikinis, "On Area Preserving Mappings of Minimal Distortion," System Theory: Modeling, Analysis, and Control, (2000).

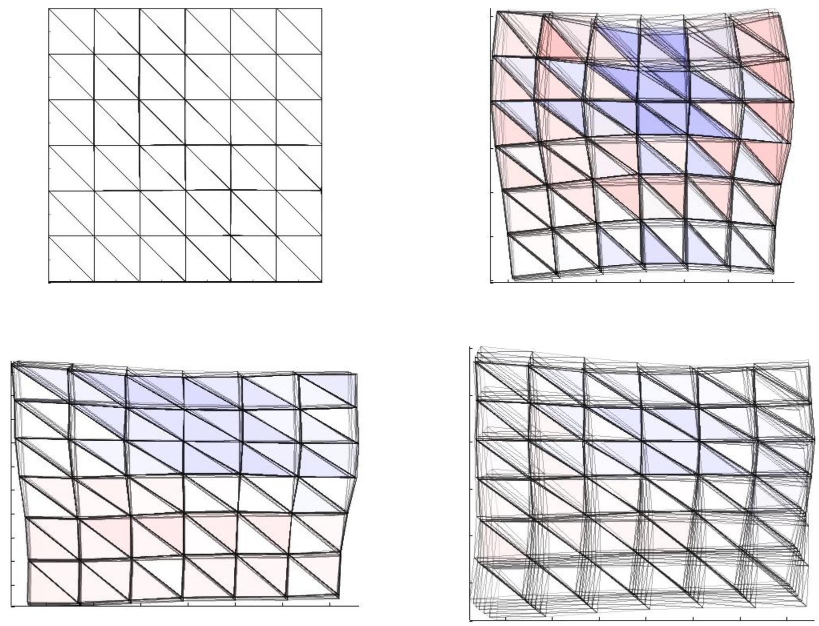

Figure 4. Views of the projected grid. Top left: original grid. Top right: registration without volume constraints. Bottom left: volume constrained deformation with elastic regularization. Bottom right: volume constrained deformation with cubic regularization $\mathcal{R}(u) = \int |\nabla u|^3$.

[20] G. Hermosillo, "Variational Methods for Multimodal Image Matching," PhD thesis, Université de Nice, France, 2002.

[21] J. B. A. Maintz and M. A. Viergever, *A survey of medical image registration*, Medical Image Analysis, **2** (1998), 1–36.

[22] M. I. Miller, A. Trouvé and L. Younes, *On the metrics and euler-lagrange equations of computational anatomy*, Annual Review of Biomedical Engineering, **4** (2002), 375–405.

[23] M. I. Miller and L. Younes, *Group actions, homeomorphisms, and matching: A general framework*, IJCV, **41** (2001), 61–84.

[24] J. Modersitzki, "Numerical Methods for Image Registration," Oxford University Press, 2004.

[25] ———, *FLIRT with rigidity – image registration with a local non-rigidity penalty*, IJCV, (2008), 153–163. 10.1007/s11263-007-0079-3.

[26] A. Neubauer and O. Scherzer, *Finite-dimensional approximation of Tikhonov regularized solutions of nonlinear ill-posed problems*, Numer. Funct. Anal. Optim., **11** (1990), 85–99.

[27] W. Peckar, C. Schnörr, K. Rohr and H. S. Stiehl, *Parameter-free elastic deformation approach for 2d and 3d registration using prescribed displacements*, J. Math. Imaging and Vision, **10** (1999), 143–162.

[28] A. Quarteroni, R. Sacco and F. Saleri, *Numerical mathematics*, vol. **37** of Texts in Applied Mathematics, Springer-Verlag, Berlin, second ed., 2007.

[29] A. Roche, *Recalage d'images médicales par inférence statistique*, PhD thesis, Université de Nice, Sophia-Antipolis, France, 2001.

[30] T. Rohlfing, J. Maurer, R. Calvin, D. A. Bluemke and M. A. Jacobs, *Volume-preserving non-rigid registration of MR breast images using free-form deformation with an incompressibility constraint*, IEEE Transactions on Medical Imaging, **22** (2003), 730–741.

[31] D. Ruan, J. A. Fessler, M. Roberson, J. Balter and M. Kessler, *2006, nonrigid registration using regularization that accommodates local tissue rigidity*, in Proc. SPIE: Medical Imaging, 2005, p. 614412.
[32] A. Trouvé, *Diffeomorphisms groups and pattern matching in image analysis*, IJCV, **28** (1998), 213–221.
[33] C. Twining and S. S. Marsland, *Constructing diffeomorphic representations for the groupwise analysis of nonrigid registrations of medical images*, IEEE Trans. Med. Imaging, **23** (2004), 1006–1020.
[34] I. Yanovsky, C. Le Guyader, A. Leow, A. Toga, P. Thompson and L. Vese, "Unbiased Volumetric Registration via Nonlinear Elastic Regularization," UCLA CAM Report 08-27, 2008.
[35] B. Zitová and J. Flusser, *Image registration methods: A survey*, Image and Vision Computing, **21** (2003), 977–1000.

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E-mail address: Christiane.Poeschl@univie.ac.at
E-mail address: modersitzki@math.uni-luebeck.de
E-mail address: Otmar.Scherzer@univie.ac.at