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A VARIATIONAL SETTING FOR VOLUME CONSTRAINED IMAGE REGISTRATION

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ABSTRACT. We consider image registration, which is the determination of a geometrical transformation that aligns points in one view of an object with corresponding points in another view. In this paper we propose constrained variational methods which aim for controlling the change of area or volume under registration transformation. We show that the method is well-posed, prove convergence of a finite element method, and present numerical examples for 3D registration.

1. INTRODUCTION

Registration is the determination of a geometrical transformation that aligns points in one view of an object with corresponding points in another view of the same or a similar object. There exist many instances particularly in medical imaging which demand for registration. Examples include the treatment verification of pre- and post-intervention images, the study of temporal series of cardiac images, and

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the monitoring of the time evolution of a contrast agent injection subject to patient motion. Another important application is the combination of information from multiple images, acquired using different modalities, like for example computer tomography (CT) and magnetic resonance imaging (MRI), a technique also known as fusion. In the last two decades, computerized non-rigid image registration has played an increasingly important role in medical imaging, see, e.g., [13, 21, 21, 24, 27, 33] and references therein.

Recent work on image registration concerns about additional features of the wanted transformation. For example, many applications require that the transformation has to be one-to-one and the question about an appropriate mathematical framework arises. Two major direction have been suggested. One approach facilitates diffeomorphic or geodesic splines; see, e.g., [5, 10, 22, 23, 30, 31]. The underlying idea is to add time as a further dimension and to establish an energy minimizing flow of corespondent particles. An additional regularization enforces that particles can not cross and as a consequence, the flow and hence the transformation is one-to-one. These techniques are of particular interest for constructing transformation groups and for performing shape analysis. However, these techniques require an additional time integration and do not provide full control on the transformation. The second approach is based on constrained optimization; see [18, 19, 29]. After appropriate discretization, one basically controls the change of volume under transformation. Introducing the displacement u with $y(x) = x + u(x)$, the constraints are based on the determinant of the Jacobian

$$C(u)(x) = \det\left(\nabla(x + u(x))\right) \quad \text{for almost all} \quad x\ .$$

Equality $C(u)(x) = 1$ as well as inequality $k(x) \le C(u)(x) \le K(x)$ constraints have been discussed in the literature [16, 19]. The choices $0 < k(x) \le K(x) < \infty$ provide lower and upper bounds for volume changes. In contrast to the diffeomorphic approach, the equality constrained approach guarantees that the volume of tissue is constant under transformation. In particular in medical imaging, this feature can be very important for some applications as for example the monitoring of tumor growth. The inequality approach enables the usage of pre-knowledge. For example, one could restrict on subregions displaying bones with $k(x) = K(x) = 1$ and on subregions displaying soft-tissue with some relaxed bounds [15, 25]. These constrained approaches are based on a so-called discretize then optimize framework and do not address existence of a minimizer in a variational setting.

In this paper, we use new variational techniques, to present existence theory for a minimizing element of the constrained optimization problem. From a mathematical point of view the work most closely related is by [9], where variational regularization methods motivated from nonlinear elasticity have been used. This model is by its nature quasi-convex. In comparison, in our approach a constrained variational method, where the constrained set is quasi-convex.

The paper is organized as follows. In Section 2 we introduce the registration set-up for this paper, then we quote some important results from the calculus of variation, that are important for this work. With this we prove the existence of minimizers of special regularization functionals in Section 4. We are also interested in the finite dimensional approximation of the minimization problem. Hence in Section 5 we explain how to approximate the constraints and the involved functionals, and show in Theorem 8 that under certain conditions, the approximated regularized solution converges to a solution of the original registration problem. At the end we give a brief outline how we implemented the area/volume preserving image registration problem, and show a simple numerical result.

2. The Registration Setup

Given are a reference image I_0 and a template image I_1, which are assumed to be compactly supported functions $I_0, I_1 : \Omega \to \mathbb{R}$, where typically $\Omega =]0,1[^d \subset \mathbb{R}^d$ and $d = 2$ or $d = 3$. Hence, $I_1(x)$ is a gray value at spatial position x and $I_0(x) = I_1(x) = 0$ for all $x \notin \Omega$. The objective is to find a displacement $u : \Omega \to \mathbb{R}^d$ such that the distance between the transformed template image $I_1(id + u)$ and the reference is small. In principle every integral based distance measure can be used; see, e.g., [20, 24, 28] for an overview. For ease of presentation, we focus on

$$\mathcal{S}(u) = \int_\Omega |I_1(x + u(x)) - I_0(x)|^2 \, dx \ . \tag{1}$$

The objective of minimizing this distance measure is known to be ill-posed (in the sense that small perturbations in the input data may lead to significant distortions in the solution) [9, 32] and hence regularization becomes inevitable. Different choices are listed in Remark 3 but we focus on the squared gradients $\mathcal{R}(u) = \int_\Omega \nabla u \cdot \nabla u \; dx = \sum \int \nabla u^i \cdot \nabla u^i$. The registration problem can thus be stated as

$$\text{minimize} \quad \mathcal{T}(u) = \mathcal{S}(u) + \alpha \mathcal{R}(u) \quad \text{subject to} \quad u \in \mathcal{A} \ , \tag{2}$$

where $\mathcal{A}$ describes one of the following sets of constraints:

$$\mathcal{A}_b^p \ := \ \{u \in W^{1,p} \mid \|u\|_{L^p} \leq b\}, \quad b < \infty \ , \tag{3}$$

$$\mathcal{A}_E^{s,p} \ := \ \left\{u \in W^{s,p} \mid C(u) = 1 \text{ a.e. in } \Omega\right\} \ , \tag{4}$$

$$\mathcal{A}_I^{s,p} \ := \ \left\{u \in W^{s,p} \mid k \leq C(u) \leq K \text{ a.e. in } \Omega\right\} \ . \tag{5}$$

The set of bound constraints $\mathcal{A}_b^p$ is very general since the data is typically given on a bounded domain and it is thus no limitation to bound the displacement by twice the diameter of the domain. *Volume preservation* is an important feature for example for tumor growth monitoring, the according transformations are collected in the sets $\mathcal{A}_E^{s,p}$ (equality constraints). The elements of the sets $\mathcal{A}_I^{s,p}$ satisfy more general *volume constraints* and enable a fine tuning of the displacement for different tissue types. Note that for the particular choices $K(x) \equiv k(x) \equiv 1$ the equality constraints are a special case of the inequality constraints.

Remark 1. *For spatial dimension $d = 1$ and a smooth and differentiable displacement u, the condition $u \in \mathcal{A}_E^{1,p}$ implies $u'(x) = 0$ for all $x \in \Omega$. Hence the only feasible transformation is a translation $y(x) = x + b$ with $b \in \mathbb{R}$. For $d = 2$, the situation is more complex. For example, any transformation $y(x) = x + u(x)^T$ with*

$u(x) = (0, g(x_1))^T$ does fulfill the constraints $C(u) = 1$, independent on the choice of g:

$$C(u) = \begin{vmatrix} 1 & 0 \\ g' & 1 \end{vmatrix} = 1 \, .$$

In general, $C(u) = 1$ leads to non-linear differential constraints since the determinant results in a polynomial of degree d in the partial derivatives of u. For example, the determinant for $d = 2$ is given by $\det(\nabla u) = u^1_x u^2_y - u^1_y u^2_x$.

3. The Variational Setup

The goal of this paper is to characterize and identify feasible choices and combinations of regularization functionals and set of constraints guaranteeing the existence of minimizing elements of problem (2). The main result in this section is given by Theorem 5. The first part of this section concentrates on existence results in general amd the second part is dedicated to the set of volume constrained functions. Theorem 4 states under which assumptions these sets are weakly closed.

For the results in the following subsection we use the concept of Carathéodory functions [8]. It is therefore convenient to rewrite the functional $\mathcal{T}$ as

$$\mathcal{T}(u) = \int f(x, u(x), \nabla u(x)) \; dx,$$

with $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d,d} \to \overline{\mathbb{R}}, \quad (x, u, A) \mapsto f(x, u, A)$. For example, choosing $\mathcal{R}(u) = \int_\Omega \nabla u : \nabla u \; dx$ results in $f(x, \xi, A) = |I_1(x + \xi) - I_0(x)|^2 + \alpha A \cdot A$. Finally, if second order derivatives are involved, we set $\overline{\mathcal{T}}(u) = \int \overline{f}(x, \xi, A, H) \; dx$, where the last component H stands for the second order terms. The following definition of a Carathéodory function extend straightforwardly to the higher order case.

Definition 1 (Carathéodory [8])**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{\infty\}$. Then f is a Carathéodory function if*

(1) *$f(x, \cdot, \cdot)$ is continuous for almost every $x \in \Omega$,*
(2) *$f(\cdot, u, A)$ is measurable in x for every $(u, A) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}$.*

3.1. **Existence Theorems in the Calculus of Variations.** We summarize conditions on f which ensure that the constrained problem (2) has a minimizing element. The following results are collected from [8].

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with a Lipschitz boundary. Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{\infty\}$ be a* Carathéodory *function satisfying the coercivity condition*

$$f(x, u, A) \geq \beta|A|^p + \gamma(x) \tag{6}$$

for almost every $x \in \Omega$, for every $(u, A) \in \mathbb{R}^d \times \mathbb{R}^{d,d}$ and for some function $\gamma \in L^1(\Omega, \mathbb{R})$, and $\beta > 0$ and $p > 1$. Assume that f is convex in A. Let $\mathcal{A} \subset u_0 + W_0^{1,p}$ be a weakly closed set of admissible functions and

$$\mathcal{T} : W^{1,p} \to \mathbb{R}\{\infty\}, \qquad u \mapsto \mathcal{T}(u) = \int_\Omega f\big(x, u(x), \nabla u(x)\big)dx\,.$$

Moreover, assume that there exists $z \in \mathcal{A}$ with $\mathcal{T}(z) < \infty$, then $\mathcal{T}(u)$ attains a minimum.

Proof. The central assumption is that the Carathéodory function f is convex with respect to A. This is a sufficient condition for $\mathcal{T}$ to be weakly lower semi-continuous in $W^{1,p}$ [8]. From this and the coercivity of f it follows that $\inf\{\mathcal{T}(u) \mid u \in u_0+W_0^{1,p}\}$ attains a minimum. The proof remains the same if one replaces $u_0+W_0^{1,p}$ by $\mathcal{A}$. Since $\mathcal{A}$ is weakly closed by assumption, the minimizing element i.e. the limit of a minimizing sequence is an element of $\mathcal{A}$. □

Remark 2. *In Theorem 1 it is assumed that the weakly closed set $\mathcal{A}$ is a subset of $u_0 + W_0^{1,p}$. Thus, the boundary values are given a fixed but unknown $u_0 \in W^{1,p}$.*

In the original proof of Theorem 1 in [8] *the condition $\mathcal{A} \subset u_0 + W_0^{1,p}$ guarantees that a minimizing sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in the L^p-norm (via the Poincaré-inequality). However, in our application all elements of $\mathcal{A}$ are bounded in the L^p-norm and it is sufficient to assume that $\mathcal{A} \subset W^{1,p}$ instead of $\mathcal{A} \subset u_0 + W_0^{1,p}$. This implies that no additional boundary conditions on the transformation have to be* *imposed. To show this, consider* $\mathcal{A}_b^p$ *for some* $b < \infty$. *Note that* $\mathcal{A}_b^p$ *is convex and closed in* $W^{1,p}$. *The mapping* $M : W^{1,p} \to \mathbb{R}$, $u \to \|u\|_{L^p}$ *is continuous. Thus* $\mathcal{A}_b^{1,p}$ *is the pre-image of* $[0, b]$ *under a continuous mapping, thus it is closed with respect to the* $W^{1,p}$*-norm. Since* $\mathcal{A}_b^p$ *is convex and closed in* $W^{1,p}$, *it is weakly closed in* $W^{1,p}$.

The previous theorem can be extended to higher order derivatives.

Theorem 2. *Let* $\overline{f} : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d\times d}$ *be Carathéodory. Assume that for each fixed* $x \in \Omega$, $u \in \mathbb{R}^d$, *and* $A \in \mathbb{R}^{d,d}$ *the mapping*

$$H \mapsto \overline{f}(x, u, A, H)$$

is convex. Moreover assume that $\overline{f}$ *satisfies the coercivity condition*

$$\overline{f}(x, u, A, H) \geq \beta |H|^p \tag{7}$$

for some $\beta > 0$ *and* $p > 1$. *Let* $\mathcal{A} \subset u_0 + W_0^{2,p}$ *be a weakly closed set of admissible functions and*

$$\overline{\mathcal{T}}(u) = \int_\Omega \overline{f}\big(x, u(x), \nabla u(x), H(u(x))\big)\, dx.$$

Assume that there exists $z \in \mathcal{A}$ *with* $\overline{\mathcal{T}}(z) < \infty$, *then* $\inf\{\overline{\mathcal{T}}(u) \mid u \in \mathcal{A}\}$ *is attained for some element of* $\mathcal{A}$.

Proof. Analogously to [8, Theorem 4.1]. □

3.2. **Weak Closedness of the constraint sets.** To show the existence of a minimizing element for the registration functional we had to assume that the sets of constraints are closed with respect to the weak topologies (see Theorems 1 and 2). Therefor we prove in the following the weak closedness of the sets introduced in the first section. In order to do so we first quote some results on weak continuity of determinants from [8].

Theorem 3 (Weak Continuity of Determinants)**.** *For the Sobolev differentiation indexes* $s = 1, 2$ *assume that* $u_k \rightharpoonup u$ *weakly in* $W^{s,p}$.

(1) *For $s = 1$ and $d = p$, it holds in the distributional sense that $C(u_k) \rightharpoonup C(u)$ weakly in $\mathcal{D}'(\Omega)$: $\forall \phi \in \mathcal{D}(\Omega)$: $\int_\Omega C(u_k(x))\phi(x)\ dx \to \int_\Omega C(u(x))\phi(x)\ dx$, where $\mathcal{D}(\Omega)$ denotes the set of compactly supported $\mathcal{C}^\infty$-functions.*

(2) *For $s = 1$ and $d < p$, $C(u_k) \rightharpoonup C(u)$ weakly in $L^{\frac{p}{d}}(\Omega, \mathbb{R})$.*

(3) *For $s = 2$ and $p < d$, $C(u_k) \to C(u)$ strongly in $L^{\frac{dp}{d-p}}(\Omega, \mathbb{R})$.*

(4) *For $s = 2$ and $p = d$, $C(u_k) \to C(u)$ strongly in $L^{\frac{q}{d}}(\Omega, \mathbb{R})$ with $p < q < \infty$.*

(5) *For $s = 2$ and $p > d$, $C(u_k) \to C(u)$ weakly in $L^{\frac{p}{d}}(\Omega, \mathbb{R})$.*

Proof. For (1), see [8, Chapter 4, Theorem 2.6]; for (2), see [11, Section 8.2.4, Lemma]; for (4), use $W^{2,p}(\Omega) \hookrightarrow W^{1,q}(\Omega), p \leq q < \infty$; then $\nabla u_k \to \nabla u$ in L^q and thus $C(u_k) \to C(u)$ in $L^{q/p}$ for $p < q$. For (3), use $W^{2,p}(\Omega) \subset W^{1,\frac{dp}{d-p}}(\Omega)$; see Compact Imbedding Theorems in [1]; then $\nabla u_k \to \nabla u$ in $L^{\frac{dp}{d-p}}$ and thus $C(u_k) \to C(u)$ in $L^{\frac{p}{d-p}}$. For 5, use item (2). □

We now prove that that the set of functions satisfying the box constraints $\mathcal{A}_I^{1,p}$, $\mathcal{A}_I^{2,p}$ — and as a special case the equality constraints $\mathcal{A}_E^{1,p}$ and $\mathcal{A}_E^{2,p}$ — are weakly closed. In the following proof, we distinguish the cases $p > d$ and $p = d$.

Theorem 4.

(1) *For $d \geq 2$ and $p \geq d$, the sets $\mathcal{A}_I^{1,p}, \mathcal{A}_E^{1,p}$ are weakly closed with respect to the $W^{1,p}$-topology.*

(2) *The sets $\mathcal{A}_I^{2,p}, \mathcal{A}_E^{2,p}$ are weakly closed with respect to the $W^{2,p}$-topology.*

Proof. For part (1) we distinguish the cases $p = d$ and $p > d$. For $p > d$, the mapping

$$M : \mathcal{A}_I^{1,p} \to L^{\frac{p}{d}}(\Omega, \mathbb{R}), \quad u \mapsto C(u)$$

is continuous with respect to the weak topology on both $W^{1,p}$ and $L^{\frac{p}{d}}(\Omega, \mathbb{R})$; see Theorem 3. Hence, the set $\mathcal{A}_I^{1,p}$ is a pre-image of the closed set $\{u \in L^{p/d} : k \leq C(u) \leq K$ a.e. in $\Omega\}$ under the weakly continuous mapping M with respect to the weak topology on $W^{1,p}$. Thus $\mathcal{A}_I^{(1,p)}$ is weakly closed.

For $p = d$, we assume $0 < K \leq B < \infty$ and define the set

$$\mathcal{B}^{1,p} := \{u \in W^{1,p} \mid \|C(u)\|_{L^\infty} \leq B\}.$$

First we prove that $\mathcal{B}^{1,p}$ is weakly closed with respect to the $W^{1,p}$-topology, then we show that the mapping $M : \mathcal{B}^{1,p} \to L^q(\Omega, \mathbb{R})$, $u \mapsto C(u)$ is continuous with respect to the weak topology on $W^{1,p}$ and the weak topology on L^q. With this we can argue as before, that $\mathcal{A}_I^{(1,p)}$ is the pre-image closed set of a weakly closed mapping and consequently it is weakly closed.

Every weak convergent sequence $(u_k)_k$ in $\mathcal{B}^{1,p}$ with weak limit u induces a sequence $c_k := C(u_k)$ in L^∞. Since $\sup_{k\in\mathbb{N}} \{\|c_k\|_{L^\infty}\} \leq B$ and according to the Alaoglu-Bourbaki-Kakutani Theorem, $(c_k)_k$ contains a weak $*$ convergent subsequence $(c_{k_i})_i$ with a weak limit $z \in L^\infty$,

$$\forall\ \phi \in L^1(\Omega, \mathbb{R}) : \quad \lim_{k\to\infty} \int_\Omega C(u_{k_i})\phi\ dx = \int_\Omega z\phi\ dx \tag{8}$$

From Theorem 3 we know that for $u_k \rightharpoonup_{W^{1,p}} u$, $C(u_k) \rightharpoonup C(u)$ weakly in $\mathcal{D}'(\Omega)$,

$$\forall\ \phi \in \mathcal{C}_0^\infty : \quad \int_\Omega C(u_k)\phi\ dx \to \int_\Omega C(u)\phi\ dx.$$

Moreover, since $\mathcal{C}_0^\infty(\Omega) \subset L^1(\Omega)$ we have $z = C(u)$ and thus $C(u) \in L^\infty(\Omega)$. Hence $\mathcal{B}^{1,p}$ is weakly closed.

Since Ω is bounded, $L^\infty(\Omega) \subset L^q(\Omega)$ and $L^{q'}(\Omega) \subset L^1(\Omega)$, for $1 < q < \infty$, where $\frac{1}{q} + 1q' = 1$. Hence for $u_k \in \mathcal{B}^{1,p}$, $C(u_k) \in L^q$ and thus (8) also holds for all $\phi \in L^{q'}$. This implies that the mapping M is weakly closed with respect to the weak topology on $W^{1,p}$ and the weak topology on L^q. Thus $\mathcal{A}_I^{1,p} \subset \mathcal{B}^{1,p}$ is the pre-image of the closed set

$$\{u \in L^q(\Omega, \mathbb{R}^d) \mid k \leq C(u) \leq K \text{ a.e. in } \Omega\}$$

under the weak continuous mapping. Hence, $\mathcal{A}_I^{1,p}$ is weakly closed.

For the seond statement we show that $\mathcal{A}_I^{2,p}$ is weakly closed.

According to Theorem 3 there exists $\delta > 0$ such that $M : \mathcal{A}_I^{2,p} \to L^{1+\delta}(\Omega, \mathbb{R}),\ u \mapsto C(u)$ is continuous with respect to the weak topologies on $W^{2,p}$ and $L^{1+\delta}(\Omega, \mathbb{R})$. The set $\mathcal{A}_I^{(2,p)}$ is thus the pre-image of the closed set

$$\{u \in L^{1+\delta} \mid k \leq C(u) \leq K \text{ a.e. in } \Omega\}$$

under the weak continuous mapping M, and hence weakly closed. □

4. Minimizing Elements for the Registration Problems

In the previous section we proved the weak closedness of the constraint sets. We are now ready to prove the existence of minimizing elements for the registration problem (2). According to Theorems 1 and 2, it remains to check the following conditions for f or $\overline{f}$ in case of higher order regularization:

(1) f or $\overline{f}$ is a **Carathéodory** function,
(2) f or $\overline{f}$ satisfies the **coercivity** condition (6) or (7), respectively,
(3) the **admissible set** of functions $\mathcal{A} \subset \mathcal{A}_b^{1,p}$ is **weakly closed** with respect to $W^{1,p}$; $\mathcal{A} \subset u_0 + W_0^{2,p}$ is weakly closed with respect to $W^{2,p}$.

Theorem 5. *Assume that I_0 and I_1 are continuous and that the sets $\mathcal{A}_b^{1,p}$ and $\mathcal{A}_I^{s,p}$ are as in* (3) *and* (5)*, respectively. For the following constrained image registration functionals exist minimizing elements:*

(1) *for $d \geq 2$, $p \geq d$, and*

$$\mathcal{T}(u) = \mathcal{S}(u) + \frac{\alpha}{p} \int |\nabla u|^p \, dx \to \min \text{ subject to } u \in \mathcal{A}_I^{1,p} \cap \mathcal{A}_b^{1,p},$$

(2) *For $d \geq 2$, $p \geq 1$, and higher order regularization*

$$\overline{\mathcal{T}}(u) = \mathcal{S}(u) + \frac{\alpha_1}{p} \int |\nabla u|^p \, dx + \frac{\alpha_2}{p} \int |H(u)|^p \, dx \to \min$$

subject to $u \in \mathcal{A}_I^{2,p} \cap \left(u_0 + W_0^{2,p}\right)$. Here $H(u)$ is the hessian of u.

Proof. Since the images I_0 and I_1 are assumed to be continuous, the associated f or $\overline{f}$ for the higher order case is a Carathéodory function.

(1) For $d \geq 2$ and $p \geq d$ we have

$$f(x, \xi, A) = (I_1(x+\xi) - I_0(x))^2 + \frac{\alpha}{p} |A|^p ,$$

thus $\frac{\alpha}{p} |\nabla u|^p \leq f(x, u, \nabla u)$, and hence f is coercive in $W^{1,p}$. Moreover, $\mathcal{A}_b^{1,p} \cap \mathcal{A}_I^{1,p}$ is weakly closed with respect to the $W^{1,p}$-norm; cf. Theorem 4. According to Theorem 2, $\mathcal{T}$ attains a minimum.

(2) For higher order regularization,

$$\overline{f}(x, \xi, A, H) = (I_1(x+\xi) - I_0(x))^2 + \alpha_1 |A|^p + \alpha_2 |H|^p.$$

Since $\frac{\alpha_2}{p} |H(u)|^p \leq \overline{f}(x, u, \nabla u, H(u))$ the coercivity condition is satisfied in $W^{2,p}$. The set of admissible functions $\mathcal{A}_I^{(2,p)} \cap (u_0 + W_0^{2,p})$ is weakly closed with respect to the $W^{2,p}$-norm; cf. Theorem 4. According to Theorem 2, $\overline{\mathcal{T}}(u)$ attains a minimum.

□

Remark 3 (Different Regularization Functionals)**.** *Theorem 5 indicates that the space dimension d determines the choice of the regularization functional $\mathcal{R}$ strongly. The limiting factor is the determinant constraint, since the determinant is a polynomial of degree d. We comment on a number of popular regularizer.*

The diffusion regularizer

$$\mathcal{R}_{\text{diff}}(u) := \frac{1}{2} \sum_{i=1..d} \left\| \nabla u^i \right\|_{L^2}^2 = \frac{1}{2} \left\| \nabla u \right\|_{L^2}^2$$

penalizes oscillating deformations and consequently leads to smooth displacement fields. $\mathcal{R}_{\text{diff}}$ is a special case of the regularizer of in Theorem 5(2): $\alpha_1 = 0$ and $p = 2$. Hence existence of a minimizer of the constrained registration functional is only guaranteed in space dimension $d = 2$. For $d = 3$ case one has to leave the Hilbert space setting, choose a regularization functional of the form $\mathcal{R}_3(u) := \left\| \nabla u \right\|_{L^3}^3$ and minimize over a subset of $W^{1,3}$.

Various registration methods use elastic regularization [2–4, 6, 14, 17]*:*

$$\mathcal{R}_{\rm elas}(u) := \sum_{i=1..d} \sum_{j=1..d} \int_\Omega \left(\frac{\lambda_1}{2} \frac{\partial\, u^i}{\partial x_i} \frac{\partial\, u^j}{\partial x_j} + \frac{\lambda_2}{4} \left(\frac{\partial\, u^i}{\partial x_j} + \frac{\partial\, u^j}{\partial x_i} \right)^2 \right) dx \ ,$$

with material parameters $\lambda_1, \lambda_2 > 0$*. Using* $\min\{\lambda_1/2, \lambda_2/4\} \int |\nabla u|^2 \le \mathcal{R}_{\rm elas}(u)$*, and Theorem 5, existence of a minimizing element for the volume preserving constraint registration functional is only guaranteed for* $d = 2$*.*

The curvature regularizer

$$\mathcal{R}_{\rm curv}(u) := \sum_{i=1..d} \int_\Omega \Delta(u^i)^2 dx \ .$$

has been introduced in [12]*: The integral might be viewed as an approximation to the curvature of the* i^{th} *component of the displacement field, therefore does penalize oscillation. Moreover affine linear transformations are contained in the kernel of* $\mathcal{R}_{\rm curv}$*, hence this regularization functional enables a direct integration of an affine linear pre-registration. Since the coercivity condition* (7) *is not satisfied, Theorem 2 cannot be applied to prove the existence of a minimizing element. A remedy is to replace* $\mathcal{R}_{\rm curv}$ *by* $\sum_{i=1..d} \left\| H(u^i) \right\|_{L^2}^2$*, where now the existence of a minimizer is guaranteed for* $d = 2$ *and* $d = 3$*. The drawback of higher order regularization is the restriction to* $u_0 + W_0^{2,2}$*, enforcing an initial computation of a meaningful* u_0*.*

5. Finite Dimensional Approximation

In this section we study a finite dimensional approximation of the minimization problem in (2). For ease of presentation we focus on the two dimensional case. First we explain how we discretize the images and the minimization functional; see Sections 5.1. The discretization of the volume box constraints is topic of Section 5.2. The main result of this section is stated in Section 5.3, where we give necessary assumptions under which the solutions of the discretized minimization problem converges to a solution the original registration problem.

5.1. **Approximation of the Minimization Functional.** The goal is to derive a multi-level representation of the images I_0, I_1 .We estimate the error of a function and its approximation only on triangles for $W^{1+\varepsilon,2}$ functions; see Theorem 6. Thus, we restrict ourselves to the case $p = 2$ and consider the following operator

$$F : W^{1,2} \to L^2(\mathbb{R}^2, \mathbb{R}), \quad u \mapsto I_1(id + u). \tag{9}$$

Assume that $I_k \in \mathcal{C}_0^1(\mathbb{R}^2, \mathbb{R})$, the components of the gradient of I_1 are bounded by some constant

$$dI_{1,\max} := \|\nabla T\|_{L^\infty} < \infty \ .$$

Here, we consider projection onto spaces of piecewise affine function such that $\|I_{k,m} - I_k\|_{L^2}^2 \le \delta_m^{I_k}$ and the transformation of the image can be described by the operator

$$F_m : W^{1,2} \to L^2(\mathbb{R}^2, \mathbb{R}), \quad u \mapsto I_{1,m} \circ (id + u). \tag{10}$$

Since we assumed that the images are continuous, the operator F is compact and satisfies a Lipschitz condition; cf. Lemma 1. This properties are exploited in Theorem 8 which states convergence to a solution.

Lemma 1. *Assume that* $I_1 \in \mathcal{C}_0^1(\mathbb{R}^2, \mathbb{R})$*, and* F, F_m *as in* (9),(10)*. Then*

(1) F *is compact,*
(2) F *is Lipschitz continuous: there exist* $\varepsilon > 0$ *and* C_F *such that* $\|F(v) - F(u)\| \le C_F \|v - u\|$,
(3) *Assume that* $0 < k \le \det(id + u) < K$*, then*

$$\|F_m(u) - F(u)\|^2 \le k^{-1}\delta_{1,m}^I \quad \textit{for all } u \in \mathcal{D}(F) \cap \mathcal{A}_I^{1,2} \ . \tag{11}$$

Proof.

(1) First we prove that F is compact. Assume therefor that we have a $\|\cdot\|_{W^{1,2}}$-bounded sequence $\{u_i\}$, which defines a sequence $\{F^i\}$ in $L^2(\mathbb{R}^d, \mathbb{R})$, where $I_1^i := F(id + u_i) = I_1 \circ (id + u_i)$. Since $I_1 \in \mathcal{C}_0^1(\mathbb{R}^2, \mathbb{R})$, F^i is bounded in

$W^{1,2}$:

$$\begin{aligned}\left\|F^i\right\|_{L^2} &\leq \|I_1(id+u_i) - I_1(id) + I_1(id)\|_{L^2} \\ &\leq \|I_1(id+u_i) - I_1(id)\|_{L^2} + \|I_1(id)\|_{L^2} \\ &\leq dI_{1,\max}\|u_i\|_{L^2} + \|I_1(id)\|_{L^2}\end{aligned}$$

and $\left\|\nabla F^i\right\|_{L^2} \leq \|\nabla I_1(id+u_i)\nabla(id+u_i)\|_{L^2} \leq dI_{1,\max}\|\nabla(id+u_i)\|_{L^2}$. Thus F^i has a weak convergent subsequence F^{i_k} in $W^{1,2}$. Using Sobolev embeddings it follows that F^{i_k} is strong convergent in the L^2-norm. Hence F is compact.

(2) For arbitrary $x, \tilde{x} \in \Omega$ we have $|I_1(x) - I_1(\tilde{x})| \leq dI_{1,\max}|x-\tilde{x}|$. Thus we get following Lipschitz-condition for F:

$$\begin{aligned}\|F(v) - F(u)\|_{L^2} &= \|I_1 \circ (id+v) - I_1 \circ (id+u)\|_{L^2} \\ &\leq dI_{1,\max}\|v-u\|_{L^2} \ .\end{aligned}$$

(3) Since we assume that $0 < k < \det(id+u) \leq K$ we can use the transformation formula and obtain

$$\begin{aligned}&k\int |I_1(x+u(x)) - I_{1,m}(x+u(x))|^2\,dx \\ \leq &\int |I_1(x+u(x)) - I_{1,m}(x+u(x))|^2 \det(\nabla(id+u)(x))dx \\ = &\int |I_1(x) - I_{1,m}(x)|^2\,dx \leq \delta_m^{I_1} \ .\end{aligned}$$

□

Setting $\mathcal{S}_m(u) := \|I_{1,m}\circ(id+u) - I_{0,m}\|^2$ and $\mathcal{R}(u) := \|\nabla u\|^2$, the discretized objective functional reads

$$\mathcal{T}_{m,n}(u_{m,n}) = \mathcal{S}_m(u_{m,n}) + \alpha\mathcal{R}(u_{m,n}) \longrightarrow \min\,. \tag{12}$$

Here, the index m is connected to the approximation of the images and the index n to the approximation of the Sobolev spaces.

5.2. **Approximation of the Constraints.** The final step is the approximation of the box constraints $\mathcal{A}_I^{1,2}$ by a sequence of finite dimensional subspaces $\mathcal{A}_n$. We aim for an approximation which allows easy handling of the determinant-constraint. In particular, we choose linear finite elements on triangles of polynomial degree one. With these the determinant of the approximation is a constant on each triangle of the triangulation.

Starting with a triangulation as shown in Figure 1, a refinement is obtained by dividing each triangle into four congruent triangles. This leads to a family of regular triangulations $\Gamma_n := (\tau_1, \cdots, \tau_{2\cdot 2^{2n}})$.

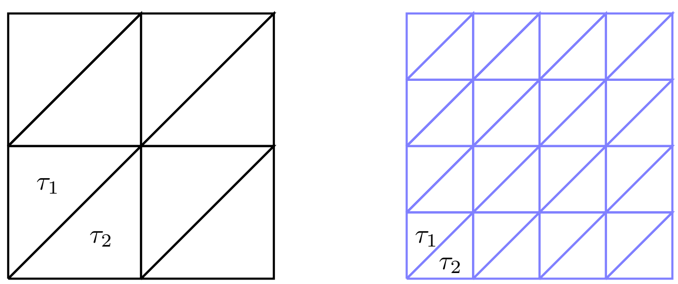


FIGURE 1. Refinement of the triangulation. Left: Γ_1, right: Γ_2.

Our displacements are elements of the following set:

$$\mathcal{U}^n := \left\{ u \in \mathcal{C}^0(\Omega, \mathbb{R}^2) \mid u_{|\tau_i} \in \Pi_1(\Omega, \mathbb{R}^2) \text{ for every } \tau_i \in \Gamma_n \right\},$$

where Π_1 is the set of polynomials of degree 1,

$$u_{|\tau_i}(x) = \begin{pmatrix} a_i^1 + b_i^1 x_1 + c_i^1 x_2 \\ a_i^2 + b_i^2 x_1 + c_i^2 x_2 \end{pmatrix} \quad \text{and hence} \quad \nabla u_{|\tau_i}(x) = \begin{pmatrix} b_i^1 & c_i^1 \\ b_i^2 & c_i^2 \end{pmatrix}.$$

By this choice of the refinement we have a nested sequence of spaces

$$\cdots \subset \mathcal{U}^n \subset \mathcal{U}^{n+1} \subset \cdots \bigcup_{m \in \mathbb{N}} \mathcal{U}^m \quad \text{and for } \omega > 0, \quad \overline{\bigcup_{n \in \mathbb{N}} \mathcal{U}^n} = W^{1+\omega,2}.$$

The challenging part is to incorporate the box constraints. For the ease of presentation, we restrict ourself to the case of constant bounds k and K and comment only on the general setting; see Remark 4.

We distinguish two cases: the integrated and the local constraints. The integrated constraints are based on a L^1-norm, and give the constraint a global nature. The bound for the discrepancy is a function of the discretization parameter h_n, i.e. basically the size of the triangles. Note that $\left\|\det(I^{2\times 2}+\nabla u_n)-1\right\|_{L^1} \leq \varepsilon(h_n)$ does not prevent $\det(I^{2\times 2}+\nabla u_n)$ to be negative in some of the triangles. Here $I^{2\times 2}\in\mathbb{R}^{2\times 2}$ denotes the identity matrix. Moreover, if h_n is small enough, $\left|\det(I^{2\times 2}+\nabla u_n(x))\right|$ can be large locally.

The reason for this choice is that under the right choice of the bound $\varepsilon(h_n)$ in depencence of h_n, (see Lemma 2) we can use the $W^{1,2}$-norm projection in order to project onto this space.

For the global L_1-norm based integrated determinant-constraint we work with

$$
\begin{aligned}
(13)\quad \mathcal{A}^1_{\varepsilon(h_n)} \quad &:= \quad \left\{u_n\in\mathcal{U}^n \mid \left\|\det(I^{2\times 2}+\nabla u_n)-1\right\|_{L^1}\leq\varepsilon(h_n)\right\}\\
&= \quad \left\{u_n\in\mathcal{U}^n \mid \frac{h_n^2}{2}\sum\left|(1+b_i^1)(1+c_i^2)-c_i^1b_i^2-1\right|\leq\varepsilon(h_n)\right\}.
\end{aligned}
$$

Here $\varepsilon(h_n)$ is a function of the discretization parameter h_n and we assume that $\lim_{h_n\to 0}\varepsilon(h_n)=0$. This assumption implies that the sets $\mathcal{A}^1_{\varepsilon(h_n)}$ are not nested. This is the reason why we do not project directly onto $\mathcal{A}^1_{\varepsilon(h_n)}$ but use the $W^{1,2}$-least square spline projection onto $\mathcal{U}^n$:

$$
(14)\qquad P_n^1: W^{1,2}\to\mathcal{U}^n,\quad u\mapsto \operatorname{argmin}_{u_n\in\mathcal{U}^n}\left\|u-u_n\right\|^2_{W^{1,2}}.
$$

As a consequence we have to specify a condition on $\varepsilon(\cdot)$ that guarantees that $P_n^1(u)\in\mathcal{A}^1_{\varepsilon(h_n)}$. This condition is given in Lemma 2.

The measure for the local determinant-constraints based on the L^∞-norm. This implies that we take the maximal change of the determinant on each triangle. A disadvantage of this choice is, that we cannot guarantee that $W^{1,2}$-norm projected functions are elements the set of functions satisfying the local determinant constraints. Hence we have to introduce an alternative projection operator. Since the corresponding sets are not convex, the projection onto them need not to be unique. For the local L_∞-norm based determinant-constraint we work with

$$\begin{aligned}(15)\qquad \mathcal{A}^{\infty}_{n,\overline{\varepsilon}} &:= \left\{u_n \in \mathcal{U}^n \mid \left\|\det(I^{2\times 2} + \nabla u_n) - 1\right\|_{L^\infty} \leq \overline{\varepsilon}\right\} \\ &= \left\{u_n \in \mathcal{U}^n \mid \left|(b_i^1 + 1)(c_i^2 + 1) - c_i^1 b_i^2 - 1\right| \leq \overline{\varepsilon}, i = 1 \ldots 2 \cdot 2^{2n}\right\},\end{aligned}$$

with a constant $\overline{\varepsilon}$. As projection operator we choose

$$(16)\qquad P_n^\infty : W^{1,2} \to \mathcal{A}^{\infty}_{n,\overline{\varepsilon}}, \quad u \mapsto \operatorname{argmin}_{u_n \in \mathcal{A}^{\infty}_{n,\overline{\varepsilon}}} \|u - u_n\|^2_{W^{1,2}}.$$

Note that a minimizing element of $u_n \to \|u - u_n\|^2_{W^{1,2}}$ over $\mathcal{A}^{\infty}_{n,\overline{\varepsilon}}$ does not have to be unique, since the sets $\mathcal{A}^{\infty}_{n,\overline{\varepsilon}}$ are not convex.

Remark 4. *Without the assumption that k, K are constant with $k = 1 - \overline{\varepsilon}$ and $K = 1 + \overline{\varepsilon}$, we would consider the sets*

$$\begin{aligned}\mathcal{A}^{\infty}_{n,k,K} :&= \left\{u_n \in \mathcal{U}^n \mid k(x) \leq \det(I^{2\times 2} + \nabla u_n) - 1 \leq K(x) \text{ f. a. e. } x \in \Omega\right\} \\ &= \{u_n \in \mathcal{U}^n \mid \min_{x\in\tau_i} k(x) \leq (b_i^1 + 1)(c_i^2 + 1) - c_i^1 b_i^2 - 1 \leq \max_{x\in\tau_i} K(x), \\ &\qquad i = 1 \ldots 2 \cdot 2^{2n}\}.\end{aligned}$$

In the following we provide a Lemma that gives a condition on the function $\varepsilon(h_n)$ needed for the definition of the global determinant constraint sets, that assures that the least squares approximation $P_n^1(u)$ in (14) is an element of the set of integrated determinant constraints. A central estimation in the proof of Lemma 2 is given by the following theorem.

Theorem 6 (Approximation of $W^{1+\omega,2}$ functions [7])**.** *Assume $\omega \geq 0$, $r \geq 1$ (degree of interpolation polynomials) and let Γ be a family of regular triangulations of Ω. Set*

$$\mathcal{U}_r^n := \left\{u \in \mathcal{C}^{r-1}(\Omega, \mathbb{R}^2) \mid u_{|\tau_i} \in \Pi_r(\Omega, \mathbb{R}^2) \text{ for every } \tau_i \in \Gamma_n\right\},$$

then

$$\inf_{u_n \in \mathcal{U}_r^n} \|u - u_n\|_{W^{1,2}} \leq \frac{C}{p^\omega} \|u\|_{W^{1+\omega,2}} h_n^{\mu-1}$$

for every $u \in W^{1+\omega,2}$*, where* $\mu := \min\{1+\omega, 1+r\}$

Lemma 2. *Let* $\omega > 0$*,* $u \in W^{1+\omega,2}$*, and* h_n *is the mesh size parameter. If*

$$\varepsilon(h_n) \geq C \left\|I^{2\times 2} + \nabla u\right\|_{L^2} \|id + u\|_{W^{1+\omega,2}} h_n^\omega ,$$

then $P_n^1(u) \in \mathcal{A}^1_{\varepsilon(h_n)}$*. The constant* C *depends on* Ω *and the regularity of the triangulation..*

Proof. We identify $\mathbb{R}^{2\times 2}$ with $\mathbb{R}^4$ such a matrix A is identified with the vector $(A_{11}, A_{12}, A_{21}, A_{22})$ and define

$$\nabla_4 := \begin{pmatrix} \frac{\partial}{\partial A_{11}} & \frac{\partial}{\partial A_{12}} \\ \frac{\partial}{\partial A_{21}} & \frac{\partial}{\partial A_{22}} \end{pmatrix}$$

Then, for $A \in \mathbb{R}^{2\times 2}$ we have

$$\begin{aligned} |\nabla_4 \det A|_{\mathbb{R}^4} &= \left| \begin{pmatrix} \frac{\partial}{\partial A_{11}}(A_{11}A_{22} - A_{12}A_{21}) & \frac{\partial}{\partial A_{12}}(A_{11}A_{22} - A_{12}A_{21}) \\ \frac{\partial}{\partial A_{21}}(A_{11}A_{22} - A_{12}A_{21}) & \frac{\partial}{\partial A_{22}}(A_{11}A_{22} - A_{12}A_{21}) \end{pmatrix} \right|_{\mathbb{R}^4} \\ &= \left| \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix} \right|_{\mathbb{R}^4} \leq |A|_{\mathbb{R}^4} . \end{aligned} \tag{17}$$

Let $A_n \in \mathbb{R}^{2\times 2}$, then Taylor-expansion gives:

$$\det(A_n) = \det(A) - \langle \nabla_4 \det(A_n), A - A_n \rangle_{\mathbb{R}^4} + O((A - A_n)^2) ,$$

which together with (17) implies

$$\begin{aligned} |\det(A_n) - \det(A)| &\leq |\langle \nabla_4 \det(A_n), A - A_n \rangle_{\mathbb{R}^4}| + O\left((A - A_n)^2\right) \\ &\leq |\nabla_4 \cdot \det(A_n)| \, |A - A_n| + O\left((A - A_n)^2\right) \\ &\leq |A_n| \, |A - A_n| + O\left((A - A_n)^2\right) . \end{aligned}$$

Hence

$$|\det(A_n) - \det(A)| = |A|\,|A - A_n| + O(|A - A_n|^2)\ .$$

For $v = id + u \in W^{1+\omega,2}$ Theorem 6 states that

$$\left\|P_n^1(v) - v\right\|_{W^{1,2}} \leq C\,\|v\|_{W^{1+\omega,2}}\,h_n^{\omega}\ .$$

Thus we get following estimate:

$$\begin{aligned}
&\left\|\det\left(\nabla P_n^1(v)\right) - \det\left(\nabla v\right)\right\|_{L^1}\\
&= \int_\Omega \left|\det\left(\nabla P_n^1(v)(x)\right) - \det\left(\nabla v(x)\right)\right| dx\\
&\leq \int_\Omega |\nabla v(x)|\left|\nabla P_n^1(v)(x) - \nabla v(x)\right| dx + O\left(\left\|\nabla P_n^1(v) - \nabla v\right\|_{L^2}^2\right)\\
&\leq \|\nabla v\|_{L^2}\left\|\nabla P_n^1(v) - \nabla v\right\|_{L^2} + O\left(\left\|\nabla P_n^1(v) - \nabla v\right\|_{L^2}^2\right)\\
&\leq \|\nabla v\|_{L^2}\,C\,\|v\|_{W^{1+\omega,2}}\,h_n^{\omega}\ .
\end{aligned}$$

Thus if $\varepsilon(h_n) \geq C\,\|v\|_{W^{1+\omega,2}}\,\|\nabla v\|_{L^2}\,h_n^{\omega}$ then $P_n^1(v) \in \mathcal{A}^1_{\varepsilon(h_n)}$ (set with integrated determinant constraints). □

A necessary ingredient for the convergence of solutions of the discretized problems converge to a solution of the inverse problem is that the projection operators converge to the identity operator.

Theorem 7. *Let* $\mathcal{A}^1_{\varepsilon(h_n)}, \mathcal{A}^\infty_{n,\overline{\varepsilon}}$ *be as in* (13), (15) *and* P_n^1, P_n^∞ *as in* (14), (16) *and* $\omega > 0$.

1. Equality constraints: *Assume that* $u \in \mathcal{A}_E^{1+\omega,2}$, $\lim_{n\to\infty}\varepsilon(h_n) = 0$ *and*

   $$\varepsilon(h_n) \geq C\,\|\nabla u\|_{L^2}\,\|u\|_{W^{1+\omega,2}}\,h_n^{\omega}\,.$$

   For $P_n^1 : \mathcal{A}^1_{\varepsilon(h_n)} \to W^{1,2}$ *we have* $\lim_{n\to\infty}\left\|P_n^1(u) - u\right\|_{W^{1,2}} = 0$.
2. Box constraints: *Assume* $u \in \mathcal{A}_I^{1+\omega,2}$ *with constant* k *and* K. *For* P_n^∞ : $\mathcal{A}^\infty_{n,\overline{\varepsilon}} \to W^{1,2}$ *we have* $\lim_{n\to\infty}\|P_n^\infty(u) - u\|_{W^{1,2}} = 0$.

Proof. For the first part we use that P_n^1 is the $W^{1,2}$-least square spline approximation of u onto $\mathcal{U}^n$ and $u \in \mathcal{A}_E^{1+\omega,2} \subset W^{1+\omega,2}$. Hence we can apply Theorem 6. Moreover Lemma 2 states that $P_n^1(u) \in \mathcal{A}^1_{\varepsilon(h_n)}$ by the choice of $\varepsilon(h_n)$.

For the second part, we denote with $\bar{}$ the closure in $W^{1,2}$ and note that $\mathcal{A}^\infty_{n,\overline{\varepsilon}} = \mathcal{U}^n \cap \mathcal{A}_I^{1,2}$. Since $\mathcal{U}^n$ is dense in $W^{1+\omega,2}$ [7], we have

$$\overline{\bigcup_{n\in\mathbb{N}} \mathcal{A}^\infty_{n,\overline{\varepsilon}}} = \overline{\bigcup_{n\in\mathbb{N}} \mathcal{U}^n \cap \mathcal{A}_I^{1,2}} = \overline{\bigcup_{n\in\mathbb{N}} \mathcal{U}^n} \cap \mathcal{A}_I^{1,2} = W^{1+\omega,2} \cap \mathcal{A}_I^{1,2} = \mathcal{A}_I^{1+\omega,2}\,.$$

Thus, for $u \in \mathcal{A}_I^{1+\omega,2}$ it holds $\lim_{n\to\infty} \|u - P_n^\infty(u)\|_{W^{1,2}} = 0$. □

5.3. **Convergence of the Approximate Solutions.** The proof that the finite dimensional solutions converge to a solution of the inverse problem is based on the following assumptions and definitions.

(1) *Assumptions on the constraints:* Let $\mathcal{A}_{con}$ denote the constraints and satisfying the volume preserving constraints, i.e $\mathcal{A}_{con} = \mathcal{A}_E^{1,2} \cap \mathcal{A}_b^2$ or $\mathcal{A}_{con} = \mathcal{A}_I^{1,2} \cap \mathcal{A}_b^2$. Assume $u \in \mathcal{A}_{con} \cap W^{1+\omega,2}$ for some $\omega > 0$. Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ denote either $\mathcal{A}^1_{\varepsilon(h_n)}$ with operator P_n^1 or $\mathcal{A}^\infty_{n,\overline{\varepsilon}}$ with operator P_n^∞.
(2) $\mathcal{D}^n := \mathcal{D}(F) \cap \mathcal{A}^n$; since $0 \in \mathcal{D}^n$, $\mathcal{D}^n \neq \emptyset$
(3) *Assumptions on the images:* For $l = 0, 1$, let $I_l \in \mathcal{C}_0^1(\mathbb{R}^2, \mathbb{R})$, with approximations $I_{l,m}$ as outlined in Section 5.1; thus, $\|I_l - I_{l,m}\|^2 = \delta_m^{I_l} \to 0$ as $m \to \infty$; set $\delta_m := k^{-1}\delta_m^{I_1} + \delta_m^{I_0}$, where k is the lower bound for the determinant constraint, as in Lemma 1.
(4) let $\alpha = \alpha(m, n)$ such that for $m, n \to \infty$ it holds

$$\alpha \to 0, \quad \delta_m/\alpha \to 0, \quad \left\|u^\dagger - P_n(u^\dagger)\right\|^2_{W^{1,2}}/\alpha \to 0. \tag{18}$$

The following result can be found in a slightly modified version in [26] and is the main result of this section:

Theorem 8. *Let $u = u^\dagger$ be a solution of the inverse problem $F(u) = I_0$. Under the above assumptions and with $\mathcal{R}(u) := \|\nabla u\|^2$, the sequence $\{u_{m_k,n_k}(\alpha_k, \delta_k)\}$*

has a convergent subsequence. The limit of every convergent subsequence is an $\mathcal{R}$-minimizing solution.

If in addition the $\mathcal{R}$-minimizing solution $u^\dagger$ is unique, then

$$\lim_{\delta_m \to 0, m \to \infty, n \to \infty} u_{m,n} = u^\dagger \ .$$

Proof. Let n be large enough, then according to Theorem 7, $P_n(u^\dagger) \in \mathcal{A}_n$. Recall that $F(u^\dagger) = I_0$, $\|I_0 - I_{0,m}\|^2 \leq \delta$. Moreover according to Lemma 1 the conditions on $I_{1,m}$ imply that $\left\|F_m(P_n(u^\dagger)) - F(P_n(u^\dagger))\right\|^2 \leq C\delta_m$. Since $u_{m,n}$ minimizes (12) we have

(19)

$$\begin{aligned} \|F_m(u_{m,n}) - I_{0,m}\|^2 + \alpha\mathcal{R}(u_{m,n}) \leq \\ &\leq \left\|F_m(P_n(u^\dagger)) - I_{0,m}\right\|^2 + \alpha\mathcal{R}(P_n(u^\dagger)) \\ &\leq 2[\left\|F_m(P_n(u^\dagger)) - F(P_n(u^\dagger))\right\|^2 + \left\|F(P_n(u^\dagger)) - F(u^\dagger)\right\|^2 + \\ &\quad \left\|F(u^\dagger) - I_{0,m}\right\|^2] + \alpha\mathcal{R}(P_n(u^\dagger)) \\ &\leq 2\left(k^{-1}\delta_m^{I_1} + C_F^2 \left\|u^\dagger - P_n(u^\dagger)\right\|^2 + \delta_m^R\right) + \alpha\mathcal{R}\left(P_n(u^\dagger)\right) \ . \end{aligned}$$

From this we obtain

$$\|F_m(u_{m,n}) - I_{0,m}\|^2 \leq \left(k^{-1}\delta_m^{I_1} + C_F^2 \left\|u^\dagger - P_n(u^\dagger)\right\|^2 + \delta_m^R\right) + \alpha \left|\mathcal{R}(P_n(u^\dagger)) - \mathcal{R}(u_{m,n}))\right| \ .$$

Taking the limit $m, n \to \infty$, we know from the assumptions on α, δ_m and P_n that $\left\|u^\dagger - P_n(u^\dagger)\right\|_L^2 \to 0$ and $\alpha \to 0$. Hence

$$\|F_m(u_{m,n}) - I_{0,m}\| \to 0 \ . \tag{20}$$

Moreover, from (19) together with Theorem 7 implying that $\mathcal{R}(P_n(u^\dagger) \to \mathcal{R}(u^\dagger)$ and the assumptions on α it follows that

$$
\begin{aligned}
&\liminf \mathcal{R}(u_{m,n}) \\
&\quad \leq \liminf \alpha^{-1}\left(k^{-1}\delta_m^{I_1} + C_F^2 \left\|u^\dagger - P_n(u^\dagger)\right\|^2 + \delta_m^R\right) + \mathcal{R}\left(P_n(u^\dagger)\right) \\
&\quad \leq \limsup \alpha^{-1}\left(k^{-1}\delta_m^{I_1} + C_F^2 \left\|u^\dagger - P_n(u^\dagger)\right\|^2 + \delta_m^R\right) + \limsup \mathcal{R}\left(P_n(u^\dagger)\right) \\
&\quad \underset{(18)}{=} 0 + \limsup \mathcal{R}(P_n(u^\dagger)) = \mathcal{R}(u^\dagger).
\end{aligned}
$$

Since $u_{m,n}$ is a $W^{1,2}$-bounded sequence, we know that $u_{m,n}$ has a weak-convergent sequence with limit $\overline{u} \in U$ which we again denote with $u_{m,n}$. Moverover taking into account (19), the fact that $u^\dagger$ is an $\mathcal{R}$-minimizing solution and the weak-lower semicontinuity of $\mathcal{R}$, we obtain

$$
\mathcal{R}\left(\overline{u}\right) \leq \liminf \mathcal{R}\left(u_{m,n}\right) \leq \mathcal{R}(u^\dagger) \; .
$$

Thus, $\mathcal{R}\left(\overline{u}\right) = \lim \mathcal{R}\left(u_{m,n}\right) = \mathcal{R}(u^\dagger)$ and $I_0 = \lim F(u_{m,n}) = F(\overline{u})$.

Moreover, (11) implies that for any subsequence u_{m_k,n_k} of $u_{m,n}$

$$
\begin{aligned}
\left\|F(u_{m_k,n_k}) - I_0\right\|^2 &\leq 2\left\|F(u_{m_k,n_k}) - F_m(u_{m_k,n_k})\right\|^2 + 2\left\|F_{m_k}(u_{m_k,n_k}) - I_0\right\|^2 \\
&\leq \underbrace{2k^{-1}\delta_{m_k}^{I_1}}_{\to 0} + \underbrace{4\left\|F_{m_k}(u_{m_k,n_k}) - I_{0,m}\right\|^2}_{\to_{(20)} 0} + \underbrace{4\delta_m^{I_0}}_{\to 0} .
\end{aligned}
$$

Taking the limit $m_k, n_k \to \infty$ such that $\delta_m \to 0$, this implies that

$$
\left\|F(u_{m_k,n_k}) - I_0\right\|^2 \to 0 \; ,
$$

Hence $\overline{u}$ is an $\mathcal{R}$-minimizing solution. □

6. Solving Registration Problem with Finite Elements

In order to solve the registration problem numerically we use a finite element approach, that is, we approximate the solutions by piecewise linear splines on triangles. Particularly, we take least squares functional $\mathcal{S}$ as in 1 to compare the reference and the deformed template image and $\mathcal{R}_2(u) := \|\nabla u\|^2$ as a regularization functional.

At the beginning of this section we introduce a penalty functional $\mathcal{P}$ to incooperate the determinant constraints. The objective is this to minimize $\mathcal{T}(u) = \mathcal{S}(u) + \alpha\mathcal{R}_2(u) + \mu\mathcal{P}(u)$. We derive the weak formulation of the corresponding differential equation and solve this system of equations with a semi explicit fixpoint iteration.

6.1. **Constraints with Penalty Functional.** The following penalty-term is used to include the determinant-constraint $u \in \mathcal{A}^{\infty}_{n,\overline{\varepsilon}}$. We define the functional

$$\mathcal{P}(u) := \int \phi_{\overline{\varepsilon}} \left(\det \left(I^{2\times 2} + \nabla u(x)\right) - 1\right) dx \,,$$

where

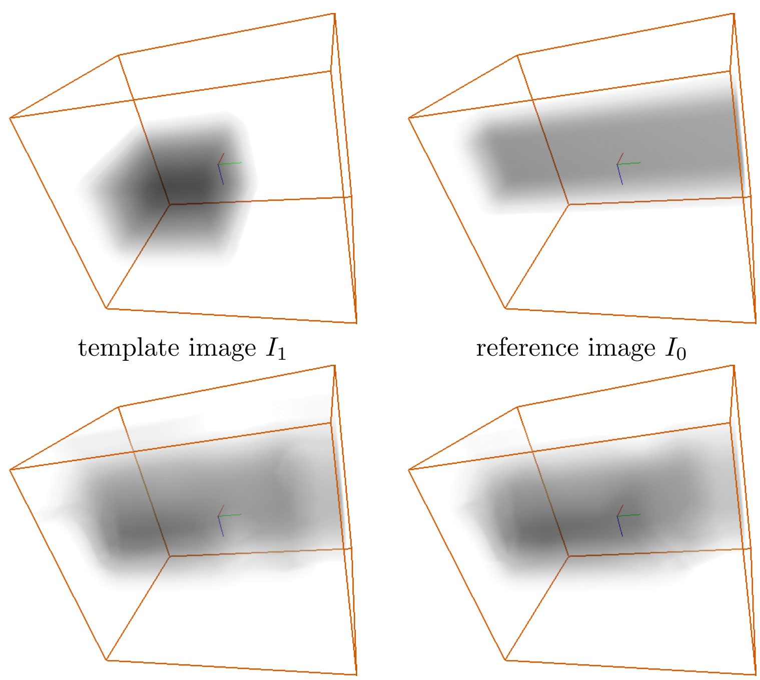
$$\phi_{\overline{\varepsilon}} : \mathbb{R} \to [0, \infty], \qquad \phi_{\overline{\varepsilon}}(c) := \begin{cases} 0 & \text{if } |c| \le \overline{\varepsilon} \\ \frac{1}{2}(c - \overline{\varepsilon})^2 & \text{if } c \ge \overline{\varepsilon} \\ \frac{1}{2}(c + \overline{\varepsilon})^2 & \text{if } c \le -\overline{\varepsilon} \end{cases} .$$

6.2. **Numerical Example.** The above scheme was implemented in C++ (2D and 3D) using the imaging2 class written by Matthias Fuchs. The imaging2 class provides an object-oriented implementation of basic mathematical objects and functions used in image processing. It includes a FEM module that provides functions to assemble the stiffness matrix and force vector for user-defined equations. The following computations are performed on a Intel Core(TM)2 Duo CPU @2.66GHz under Fedora.

The results for simplified examples are shown in Table 6.2 and Figures 6.1 and 6.1. For a stable solution of the constrained image registration problem we first calculate 150 iterations with $\mu = 0$, then we start increasing μ in each step.

TABLE 1. Parameters of the example shown in Figure 6.1.

	unconstrained	constrained
regularization parameter: α	0.8	0.8
step size parameter: β	0.5	1.0
penalty parameter: μ	0.0	0.1
box parameter: ε	-	0.2
error: $\|I_1(u+id) - I_0\|$	1.24	1.62
	0.020 %	0.025 %
$\min(\det(id+u))$	-0.20	0.35
$\max(\det(id+u))$	2.49	1.63
number of iteration steps:	200	800
computation time	40s	80s



template image I_1 — reference image I_0

$I_1(u+id)$ unconstrained solution — $I_1(u+id)$ constrained solution

FIGURE 2. Image size: $6 \times 8 \times 7$, Template box: lower left corner $(0, 1, 2)$ upper right corner at $(4, 5, 5)$, reference box: lower left corner at $(1, 0, 1)$, upper right corner at $(3, 8, 4)$.

7. CONCLUSIONS

In this paper we have investigated the existence of minimizing elements of special registration functionals. One motivation for studying these is due our previous

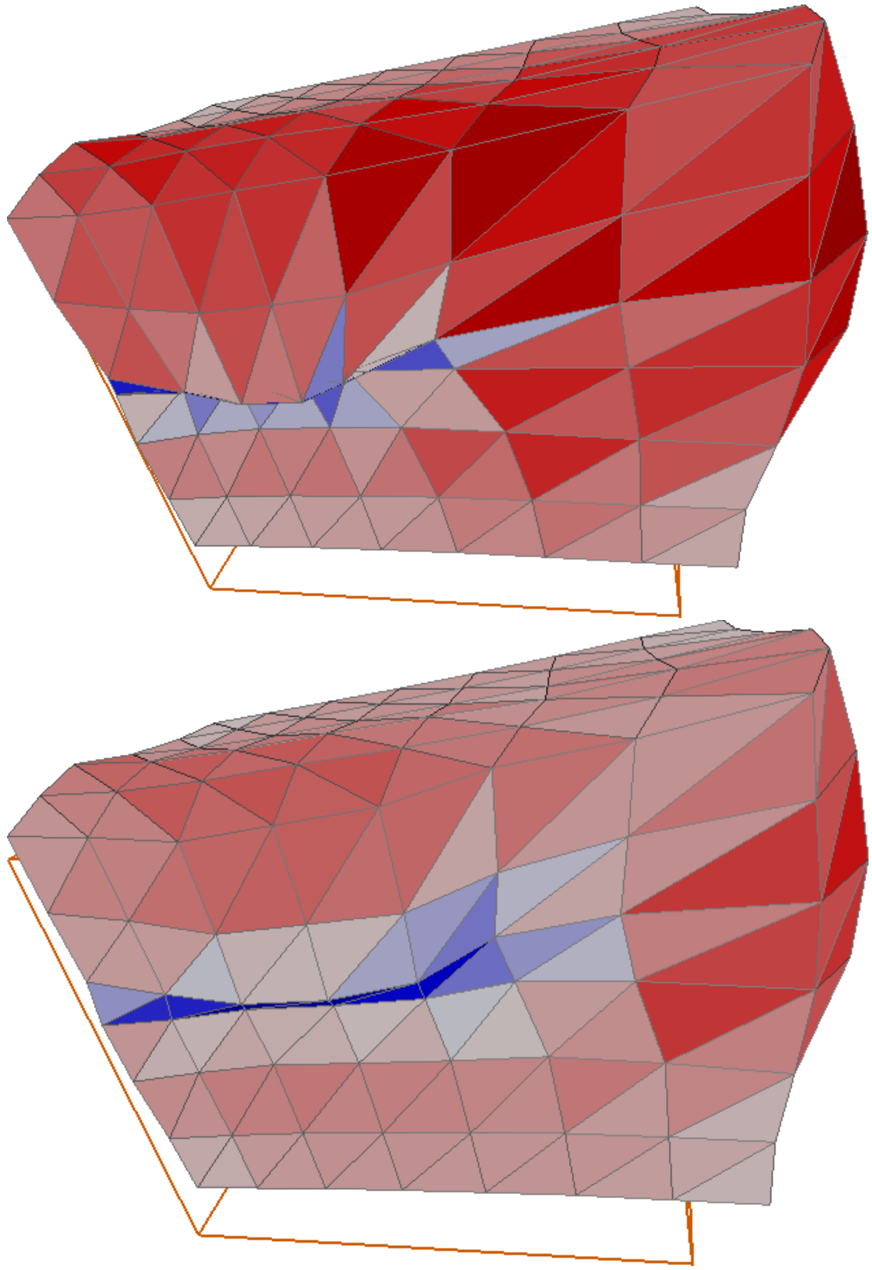


Figure 3. Deformed grid. Top image: without volume preserving constraints, the supports of the elements intersect each other. The color indicates the volume change (red - increase in volume, blue decrease in volume.

work, where we introduced numerical methods for volume preserving image registration [17]. Here we used variational techniques to prove the existence of minimizers of the registration functional.

Moreover we clarified the difficulties caused by the discretization of the area preserving constraints and proposed two ways to approximate the set of area constrained transformations. At the end we proposed a numerical scheme to implement area/volume-constrained 2D/3D-registration using finite elements.

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