IN-64
DATE OVERRIDE
43021
P-8

Optimal Chebyshev Polynomials on Ellipses in the Complex Plane

Bernd Fischer
Roland Freund

February, 1989

Research Institute for Advanced Computer Science
NASA Ames Research Center

RIACS Technical Report 89.5

NASA Cooperative Agreement Number NCC 2-387

Research Institute for Advanced Computer Science

Optimal Chebyshev Polynomials on Ellipses in the Complex Plane

Bernd Fischer
Roland Freund

February, 1989

Research Institute for Advanced Computer Science
NASA Ames Research Center

RIACS Technical Report 89.5

NASA Cooperative Agreement Number NCC 2-387

Optimal Chebyshev Polynomials on Ellipses in the Complex Plane

Bernd Fischer
Department of Computer Science
Stanford University
Stanford, CA 94305

and

Inst. fuer Angew. Mathematik
Universitaet Hamburg
D-2000 Hamburg 13, F.R.G.

Roland Freund
RIACS, Mail Stop 230-5
NASA Ames Research Center
Moffett Field, CA 94035

and

Inst. fuer Angew. Mathematik
und Statistik
Universitaet Wuerzburg
D-8700 Wuerzburg, F.R.G.

Abstract

The design of iterative schemes for sparse matrix computations often leads to constrained polynomial approximation problems on sets in the complex plane. For the case of ellipses, we introduce a new class of complex polynomials which are in general very good approximations to the best polynomials and even optimal in most cases.

The first author was supported by the German Research Association (DFG), the second by Cooperative Agreement NCC 2-387 between the National Aeronautics and Space Administration (NASA) and the Universities Space Research Association (USRA).

§1. Introduction

We consider complex Chebyshev approximation problems of the type

$$E_n(r,c) = \min_{p\in\Pi_n:\, p(c)=1} \|p\|_{\mathcal{E}_r}, \quad \|p\|_{\mathcal{E}_r} := \max_{z\in\mathcal{E}_r} |p(z)|. \tag{1}$$

Here Π_n denotes the space of all complex polynomials of degree at most n, $\mathcal{E}_r$ is any ellipse with foci ± 1 and semi-axes $(r \pm r^{-1})/2$, $r > 1$, and $c \in \mathbf{C} \setminus \mathcal{E}_r$. It will be convenient to express c as a point on the boundary $\partial\mathcal{E}_R$ of the ellipse $\mathcal{E}_R$, $R > r$, i.e. $c = c(R,\gamma) = ((R + R^{-1})\cos(\gamma) + \mathrm{i}(R - R^{-1})\sin(\gamma))/2$, $\gamma \in [0, 2\pi)$. Since Haar's condition is satisfied, there always exists a unique optimal polynomial $p_n(z; r, c)$ of (1).

Problems (1), in general with $\mathcal{E} \subset \mathbf{C}$ any compact set instead of $\mathcal{E}_r$, arise in numerical linear algebra. E.g. the design of iterative methods for the solution of large sparse non-Hermitian linear systems $Ax = b$ with best possible convergence rates [2], the computation of optimal polynomial preconditioners for conjugate gradient type algorithms for $Ax = b$ [7], or the acceleration of eigenvalue methods for A [6] all lead to problems of this type. However, for arbitrary sets $\mathcal{E}$ the optimal polynomials are in general not known explicitly and therefore the methods are usually based on polynomials which are only asymptotically optimal. A popular choice for the set $\mathcal{E}$ are ellipses, and then the scaled Chebyshev polynomials $t_n(z; c) := T_n(z)/T_n(c)$ are used as approximations to the optimal polynomials of (1) [4, 6]. Clayton [1] showed that even $t_n(z; c) \equiv p_n(z; r, c)$ if c is real, and in general t_n is nearly optimal for (1) as long as n is large. However, in some of the applications we mentioned, polynomials with small degree are used and typically the distance between c and $\mathcal{E}_r$ is small. Depending on the position of c on $\partial\mathcal{E}_R$, $\|t_n(z; c)\|_{\mathcal{E}_r} > 1$ can occur, and then t_n yields no useful approximation (cf. Example 1 given below).

In this note, we introduce a new class of asymptotically optimal polynomials q_n for Problem (1) which always satisfy $\|q_n(z; c)\|_{\mathcal{E}_r} \le \|t_n(z; c)\|_{\mathcal{E}_r}$ and $\|q_n(z; c)\|_{\mathcal{E}_r} < 1$. Moreover, they are even optimal in most cases.

§2. Results

The q_n are defined by

$$q_n(z; c) = \frac{T_n(z) + \alpha_n}{T_n(c) + \alpha_n}, \quad \alpha_n = 2\mathrm{i}\frac{\sin(n\gamma)}{(R^n - R^{-n})}. \tag{2}$$

Here α_n is the solution of the extremal problem

$$M_n(r,c) = \min_{\alpha\in\mathbf{C}} \max_{z\in\mathcal{E}_r} \left| \frac{T_n(z) + \alpha}{T_n(c) + \alpha} \right|. \tag{3}$$

We summarize the important properties of $q_n(z; c)$ in the following

Theorem 1. [3]

(a) $q_n(z;c)$ *has precisely* $2n$ *extremal points* z_j, $j = 1, 2, \ldots, 2n$, *on* $\partial\mathcal{E}_r$ *with* $\|q_n(z;c)\|_{\mathcal{E}_r} = M_n(r,c) = (r^n + r^{-n})/(R^n + R^{-n})$.

(b) There exists a number $R_0(n,r)$ *such that* $q_n(z;c) \equiv p_n(z;r,c)$ *for all* $c \in \partial\mathcal{E}_R$ *with* $R \geq R_0(n,r)$.

(c) Let $c \in \partial\mathcal{E}_R$ *be such that* $R > r(9r^4 - 1)/(r^4 - 1)$. *Then, there exists an integer* $n_0(r,R)$ *such that* $q_n(z;c) \equiv p_n(z;r,c)$ *for all* $n \geq n_0(r,R)$.

Discussion: Supported by numerical tests (c.f. Example 2), we conjecture that (c) is true for arbitrary $R > r > 1$. $\|q_n(z;c)\|_{\mathcal{E}_r}$ does not depend on the position of c on $\partial\mathcal{E}_R$ and $\|q_n(z;c)\|_{\mathcal{E}_r} \leq \|t_n(z;c)\|_{\mathcal{E}_r}$, where equality holds iff $\sin(n\gamma) = 0$, e.g. for $c \in \mathbb{R}$ (cf. Example 1). The proof of Theorem 1 is based on the following characterization [5]: $q_n(z;c) \equiv p_n(z;r,c)$ iff the linear system

$$\sum_{j=1}^{2n} \sigma_j \overline{q_n(z_j;c)}(z_j - c)p(z_j) = 0 \quad \text{for all } p \in \Pi_{n-1} \tag{4}$$

has a nontrivial and nonnegative solution. See [3] for the explicit solution of (4).

Example 1. *We compare* $\|q_n(z;c)\|_{\mathcal{E}_r}$ *(continous curve) and* $\|t_n(z;c)\|_{\mathcal{E}_r}$ *(dashed curve) where* $r = 1.1$, $R = 1.2$ *for* $\gamma \in [0,\pi]$ *and* $n = 3, 4$ *(cf. Figure 1).*

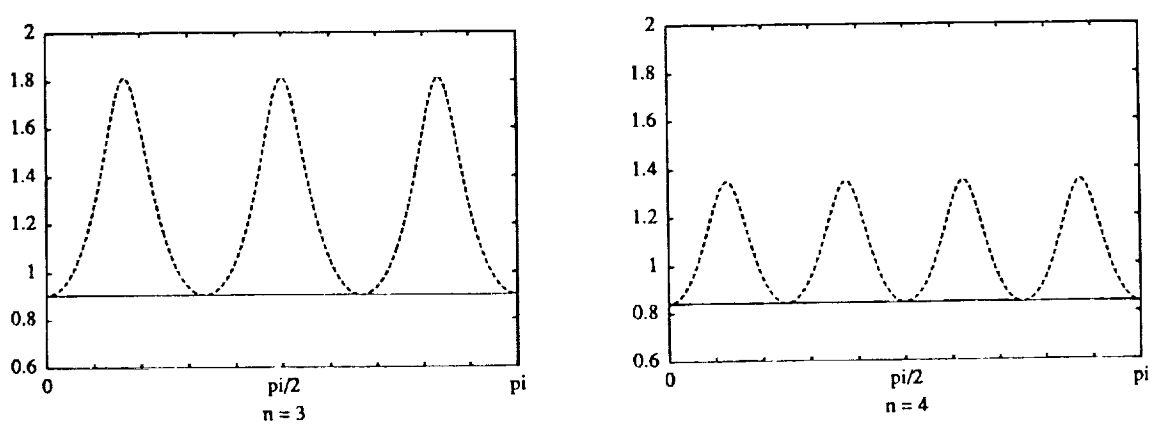


Figure 1. Maximum norm of q_n and t_n

The nontrivial solutions of (4) always lead to a lower bound for the minimal deviation of problem (1), which is sharp in a certain sense:

Theorem 2. *Let* σ_j, $j = 1, 2, \ldots, 2n$, *be any nontrivial real solution of (4), normalized such that* $\sum_{j=1}^{2n} |\sigma_j| = 1$, *then*

$$L_n(r,c) = \frac{1}{M_n(r,c)} \Big| \sum_{j=1}^{2n} \sigma_j q_n(z_j;c) \Big| \;\leq\; E_n(r,c),$$

where equality holds iff $q_n(z;c) \equiv p_n(z;r,c)$.

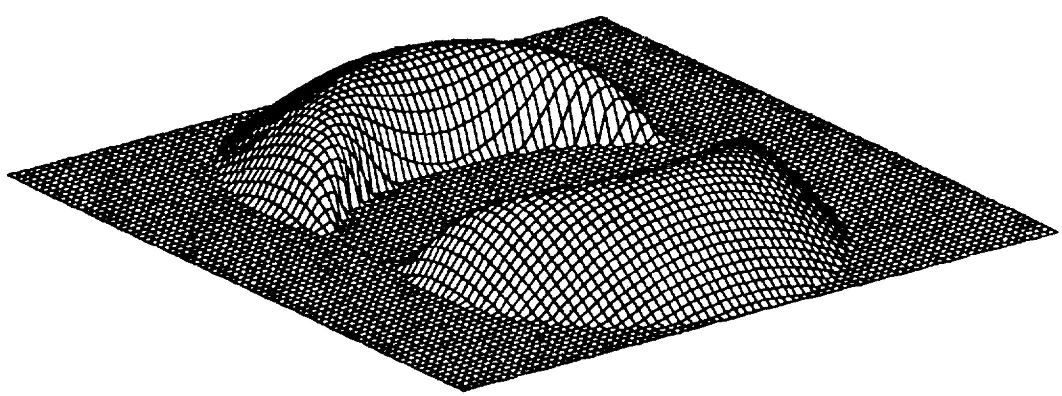
Proof: Let $p \in \Pi_{n-1}$. From (4) we obtain

$$\left|\sum_{j=1}^{2n} \sigma_j q_n(z_j;c)\right| = \left|\sum_{j=1}^{2n} \sigma_j \overline{q_n(z_j;c)}\right| = \left|\sum_{j=1}^{2n} \sigma_j \overline{q_n(z_j;c)}(1-(z_j-c)p(z_j))\right|$$
$$\leq \left(M_n(r,c)\sum_{j=1}^{2n}|\sigma_j|\right) \quad \|1-(z-c)p(z)\|_{\mathcal{E}_r},$$

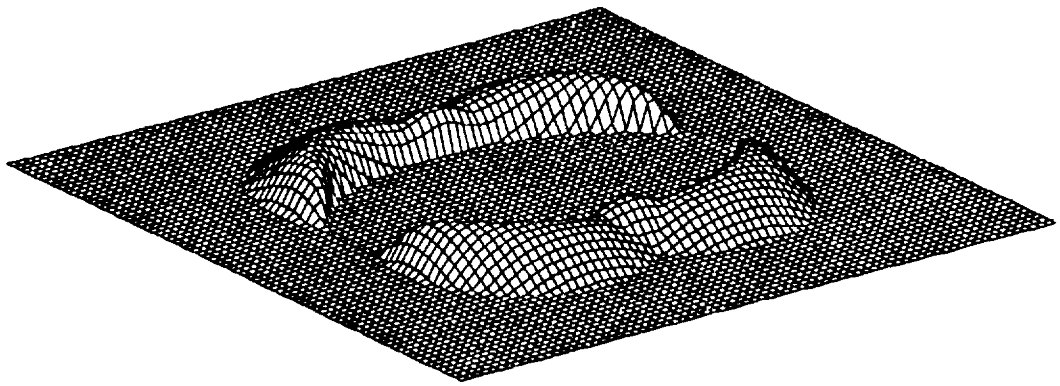
and the result follows. ■

We illustrate that q_n is in general nearly optimal in the following

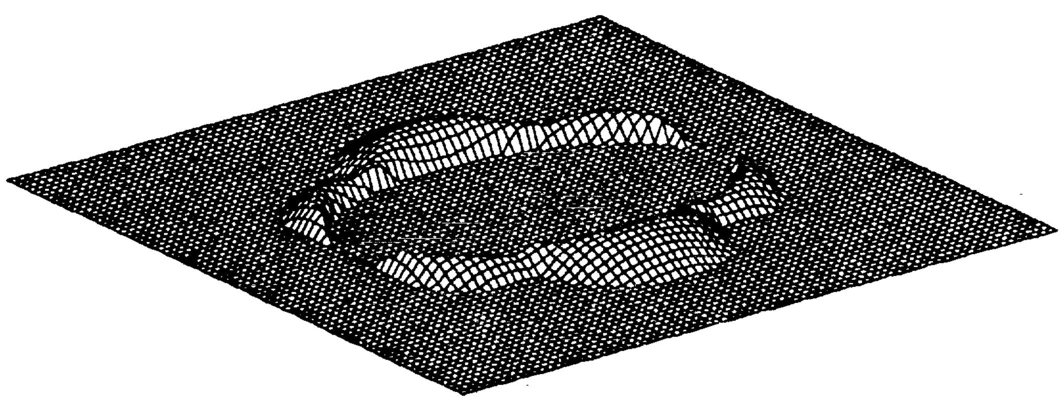
Example 2. *In this example we compute the relative deviation* $D_n(r,c) = (M_n(r,c) - L_n(r,c))/M_n(r,c)$ *where* $r = 2$ *for* $c \in [-2.1, 2.1] \times [-i2.1, i2.1]$ *(here* $D_n(r,c) := 0$ *if* $c \in \mathcal{E}_r$*) and* $n =$ 2, 3, 4, 5. *Note that* $D_n(r,c) = 0$ *if* $q_n(r,c)$ *is optimal (cf. Figure 2). We obtain* $\max_{c\in C} D_n(2,c) <$ *0.1024, 0.0498, 0.0336, 0.0210 for* $n =$ 2, 3, 4, 5 *resp.*



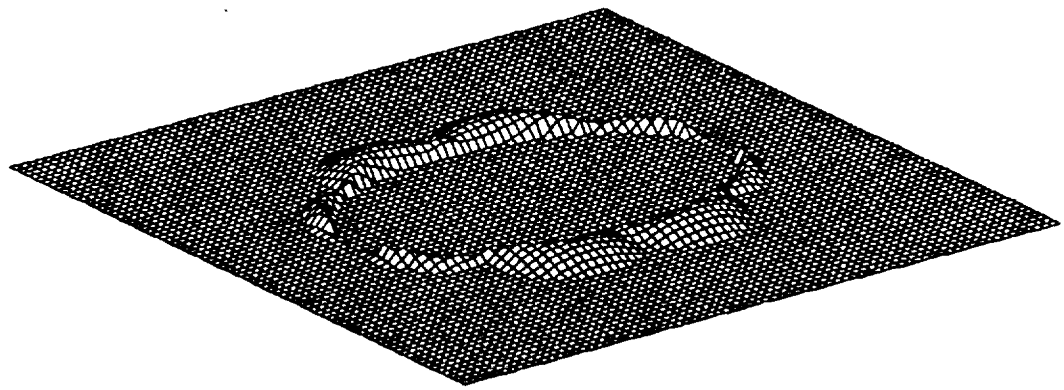
r = 2, n = 2



r = 2, n = 3



r = 2, n = 4



r = 2, n = 5

Figure 2. *Relative deviation of* q_n

References

1. Clayton, A. J., Further results on polynomials having least maximum modulus over an ellipse in the complex plane, U.K.A.E.A. Memorandum - AEEW - M 348, 1963.
2. Eiermann, M., W. Niethammer, and R. S. Varga, A study of semiiterative methods for nonsymmetric systems of linear equations, Numer. Math. **47** (1985), 505-533.
3. Fischer, B. and R. Freund, On the constrained Chebyshev approximation Problem on ellipses, J. Approx. Th. , to appear.
4. Manteuffel, T. A., The Tchebychev iteration for nonsymmetric linear systems, Numer. Math. **28** (1977), 307-327.
5. Rivlin, T. J. and H. S. Shapiro, A unified approach to certain problems of approximation and minimization, J. Soc. Indust. Appl. Math. **9** (1961), 670-699.
6. Saad, Y., Chebyshev acceleration techniques for solving nonsymmetric eigenvalue problems, Math. Comp. **42** (1984), 567-588.
7. Saad, Y., Krylov subspace methods on supercomputers, SIAM J. Sci. Stat. Comp., to appear.