Inverse Scale Space Iterations for Non-Convex Variational Problems Using Functional Lifting


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This work was supported through DFG grant LE 4064/1-1 “Functional Lifting 2.0: Efficient Convexifications for Imaging and Vision” and NVIDIA Corporation.

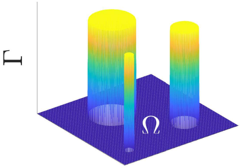
Eighth International Conference on Scale Space and Variational Methods in Computer Vision (SSVM)
May 17th 2021

Notation

(Non-convex) Variational Problem

$$F(u) := \underbrace{\int_\Omega \rho(x, u(x))\, dx}_{\text{data term } H(u)} + \underbrace{\int_\Omega \eta(\nabla u(x))\, dx,}_{\text{regularizer } J(u)}$$

- $u : \Omega \to \Gamma$ sufficiently smooth
- $\Omega \subset \mathbb{R}^d$ open and bounded image domain
- $\Gamma \subset \mathbb{R}$ compact range (label space)
- $\eta : \mathbb{R}^d \to \mathbb{R}$ non-negative and convex
- $\rho : \Omega \times \Gamma \to \overline{\mathbb{R}}$ proper, non-negative and possibly non-convex



Applications (image reconstruction, segmentation, motion estimation): Aubert and Kornprobst 2006, Scherzer et al. 2009

Bregman Iteration

Consider the denoising problem:

$$F(u) = \int_\Omega \frac{\lambda}{2} \left(u(x) - f(x)\right)^2 dx + J(u).$$

With input data f and some convex, absolute one-homogeneous regularizer J.

Bregman Iteration [Osher et al. 2005]

Initialize $p_0 = 0$ and repeat for $k = 1, 2, \dots$

$$u_k \in \arg\min_u \left\{ H(u) + J(u) - \langle p_{k-1}, u \rangle \right\}$$

$$p_k \in \partial J(u_k)$$

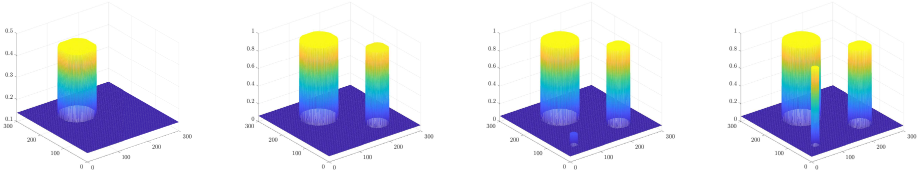
Continuous limit leads to the Inverse Scale Space Flow:

- Non-linear decompositions of the input
- Non-linear filters for the input

Bregman Iteration: Osher et al. 2005; Inverse Scale Space Flow: Burger et al. 2006, Burger et al. 2015, Burger et al. 2016, Gilboa et al. 2016

Example & Outlook

Bregman iteration on Rudin-Osher-Fatemi (convex)



Implementation with prost: Möllenhoff et al. 2016

Related Work

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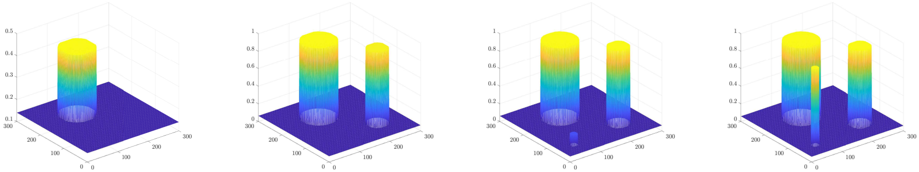
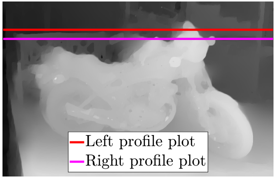
	Bregman iteration	ISS, non-linear filters	convex data term	non-convex data term	lifting
Osher et al. 2005	✓		✓		
Burger et al. 2006, 2015, 2016	(✓)	✓	✓		
Höltgen 2016	✓			(optical flow)	
Alberti et al. 2003			(✓)	✓	✓
Pock et al. 2010			(✓)	✓	✓
Vogt 2019			(✓)	✓	✓
Möllenhoff et al. 2015, 2017			(✓)	✓	(sublabel-accurate)
Lifted Bregman	✓	**?**	✓	✓	✓

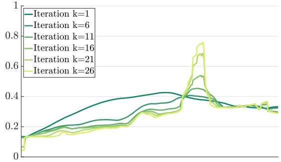
Example & Outlook

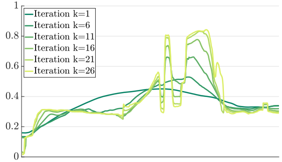
Rudin-Osher-Fatemi (convex)



Stereo-Matching (non-convex)







Implementation with prost library: Möllenhoff et al. 2016; Middlebury Stereo Datasets: Scharstein et al. 2014

Lifting Approach

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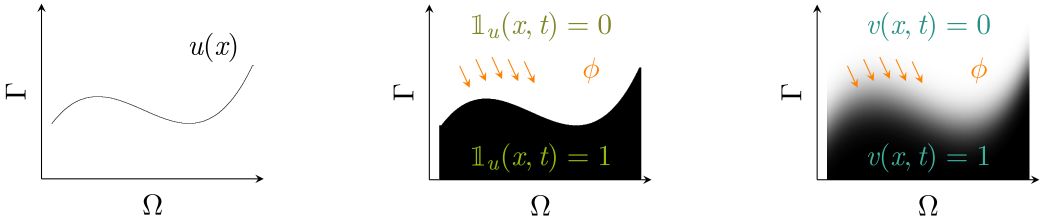
Calibration-based Lifting [Alberti et al. 2003, Pock et al. 2010]

(O) Original Problem: $\inf_{u \in W^{1,1}} F(u), \quad F(u) := \int_\Omega \rho(x, u(x)) + \eta(\nabla u(x))\, dx$

(E) Embedding: $\inf_{\mathbb{1}_u,\, u \in W^{1,1}} \mathcal{F}(\mathbb{1}_u), \quad \mathcal{F}(\mathbb{1}_u) := \sup_{\phi \in \mathcal{K}_{\rho,\eta}} \int_{\Omega \times \mathbb{R}} \langle \phi, D\mathbb{1}_u \rangle$

(C) Convexification: $\inf_{v \in C} \mathcal{F}(v)$



Calibration-based Lifting: Alberti et al. 2003, Pock et al. 2010, Vogt 2019

Lifting Approach

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(C) Convexification: $\inf_{v \in C} \mathcal{F}(v)$

Global minimizers [Pock et al. 2010 (Thm. 3. 1)]

Let v^* be a global minimizer in (C). Then for any $s \in [0, 1)$ the characteristic function $\mathbb{1}_{\{v^* > s\}}$ is a global minimizer in (E). Furthermore, the function $\mathbb{1}_{\{v^* > s\}}$ is the characteristic of the subgraph of a minimizer of (O).

Lifting Approach - Sketch

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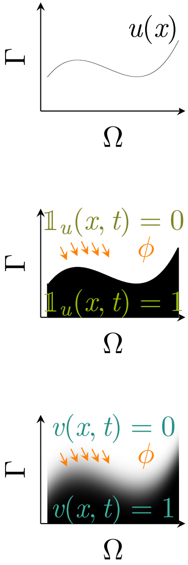
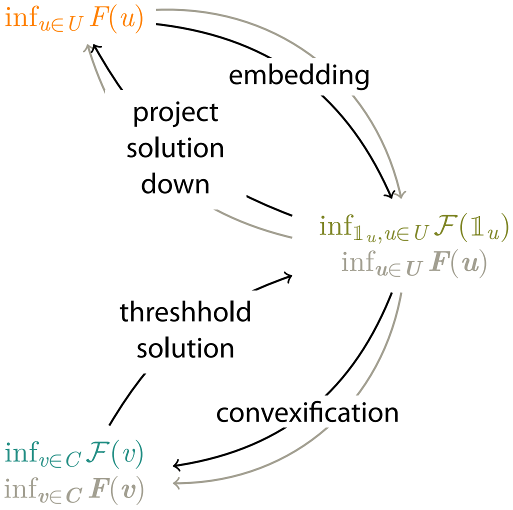


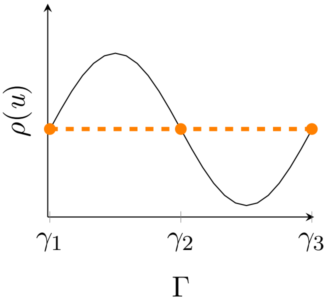


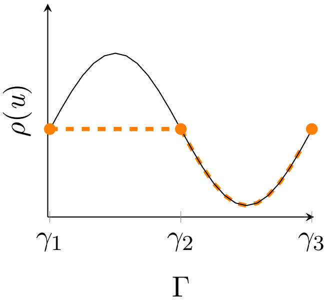
continuous domain Ω, discretized range Γ

Lifting Approach - Discretization

State of the art for $J(u) = \text{TV}(u)$: sublabel-accurate discretization

- Relies on convexity and one-homogeneity of TV
- Data term and regularizer can be lifted separately
- Results in piecewise convex approximation of the data term





Sublabel-Accurate Discretization: Möllenhoff et al. 2015, Möllenhoff and Cremers 2017

Lifting Approach - Discretization

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Notation needed for the talk:

- Discretize the range Γ by choosing $\ell + 1$ labels:

$$\gamma_1 < \ldots < \gamma_{\ell+1}, \quad \Gamma = [\gamma_1, \gamma_{\ell+1}]$$

- Denote the (sublabel-accurate discretized) lifted problem as

$$\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{H}(\boldsymbol{u}) + \boldsymbol{J}(\boldsymbol{u})$$

Sublabel-Accurate Discretization: Möllenhoff et al. 2015, Möllenhoff and Cremers 2017

Lifted Bregman Iteration

Lifted Bregman Iteration [Bednarski and Lellmann 2021]

Initialize $\boldsymbol{p}_0 = 0$ and repeat for $k = 1, 2, \ldots$

$$\boldsymbol{u}_k \in \arg\min_{\boldsymbol{u}} \left\{ \boldsymbol{H}(\boldsymbol{u}) + \boldsymbol{J}(\boldsymbol{u}) - \langle \boldsymbol{p}_{k-1}, \boldsymbol{u} \rangle \right\}$$

$$\boldsymbol{p}_k \in \partial \boldsymbol{J}(\boldsymbol{u}_k)$$

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$$\boldsymbol{p}_k \in \partial \boldsymbol{J}(\boldsymbol{u}_k)$$

Original Bregman iteration:
$u_1^* \in \arg\min_u \{H(u) + J(u)\}$ $\longleftrightarrow$

Lifted Bregman iteration:
$\boldsymbol{u}_1^* \in \arg\min_{\boldsymbol{u}} \{\boldsymbol{H}(\boldsymbol{u}) + \boldsymbol{J}(\boldsymbol{u})\}$

Solutions u_k^* and $\boldsymbol{u}_k^*$ are equivalent, if the projection of $\boldsymbol{u}_k^*$ given by

$$\tilde{u}_k^*(x) := \gamma_1 + (\gamma_2 - \gamma_1, \ldots, \gamma_{\ell+1} - \gamma_\ell)\,\boldsymbol{u}_k^*(x)$$

gives a solution of the original problem, i. e. $u_k^* = \tilde{u}_k^*(x)$

Lifted Bregman Iteration

Lifted Bregman Iteration [Bednarski and Lellmann 2021]

Initialize $\boldsymbol{p}_0 = 0$ and repeat for $k = 1, 2, \ldots$

$$\boldsymbol{u}_k \in \arg\min_{\boldsymbol{u}} \left\{ \boldsymbol{H}(\boldsymbol{u}) + \boldsymbol{J}(\boldsymbol{u}) - \langle \boldsymbol{p}_{k-1}, \boldsymbol{u} \rangle \right\}$$

$$\boldsymbol{p}_k \in \partial \boldsymbol{J}(\boldsymbol{u}_k)$$

Original Bregman iteration:

$u_1^* \in \arg\min_u \{H(u) + J(u)\}$

$u_2^* \in \arg\min_u \{H(u) + J(u) - \langle p_1, u \rangle\}$

$\longleftrightarrow$

$\overset{?}{\longleftrightarrow}$

Lifted Bregman iteration:

$\boldsymbol{u}_1^* \in \arg\min_{\boldsymbol{u}} \{\boldsymbol{H}(\boldsymbol{u}) + \boldsymbol{J}(\boldsymbol{u})\}$

$\boldsymbol{u}_2^* \in \arg\min_{\boldsymbol{u}} \{\boldsymbol{H}(\boldsymbol{u}) + \boldsymbol{J}(\boldsymbol{u}) - \langle \boldsymbol{p}_1, \boldsymbol{u} \rangle\}$

Next: sufficient condition on the subgradients p_{k-1} and $\boldsymbol{p}_{k-1}$ such that the iterates u_k^* and $\boldsymbol{u}_k^*$ are equivalent.

Equivalence

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Proposition 2 [Bednarski and Lellmann 2021]

Assume that the minimization problems in the original Bregman iteration have unique solutions. Assume that the solutions in the lifted Bregman iteration $(\boldsymbol{u}_k^*)$ are sublabel-integral.

If at every point x the chosen subgradients $p_{k-1} \in \partial J(u_{k-1})$ and $\boldsymbol{p}_{k-1} \in \partial \boldsymbol{J}(\boldsymbol{u}_{k-1})$ satisfy

$$\boldsymbol{p}_{k-1}(x) \quad = \quad p_{k-1}(x) \left(\gamma_2 - \gamma_1, \quad \ldots, \quad \gamma_{l+1} - \gamma_l\right)^\top,$$

then the solutions of the lifted Bregman iteration $(\boldsymbol{u}_k^*)$ and the original Bregman iteration (u_k^*) are equivalent.

Proof: Relies on the fact that the lifted representation of $H(u) = H_1(u) - H_2(u)$ for $H_2(u) = \langle p, u \rangle$ is $\boldsymbol{H}(\boldsymbol{u}) = \boldsymbol{H}_1(\boldsymbol{u}) - \boldsymbol{H}_2(\boldsymbol{u})$.

$\Rightarrow$ Proposition applies to deconvolution/ denoising problem with anisotropic TV

Results

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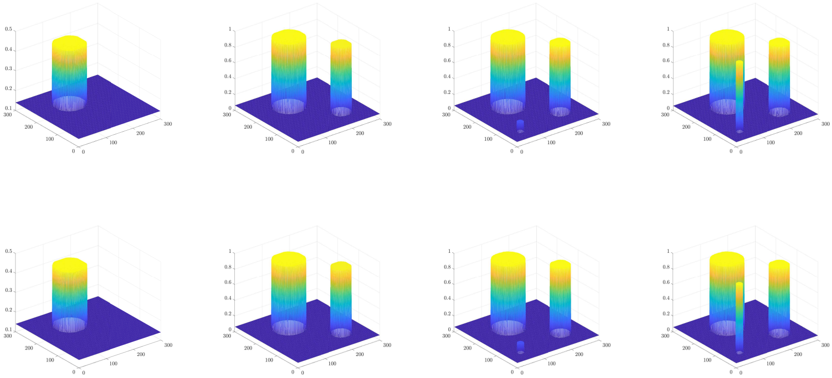
Rudin-Osher-Fatemi:

$$F(u) = \int_\Omega \underbrace{\frac{\lambda}{2}(u(x) - f(x))^2}_{\rho(x, u(x))} \, dx + \mathsf{TV}(u)$$

Bregman

Lifted Bregman transformed subgradients



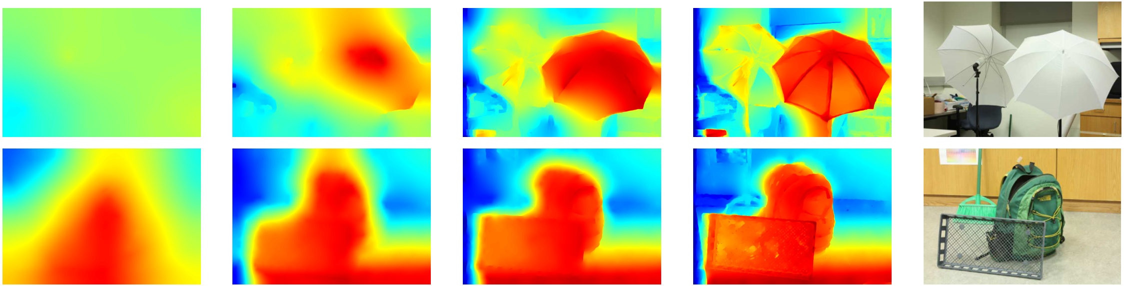
k = 1 | k = 2 | k = 3 | k = 4

Implementation with prost library: Möllenhoff et al. 2016

Results

Stereo Matching:

$$F(u) = \int_{\Omega} \underbrace{\int_{W(x)} \sum_{d=1,2} h(\partial_{x_d} I_1((y_1, y_2 + u(x))) - \partial_{x_d} I_2((y_1, y_2)))\, dx}_{\rho(x,u(x))} + \text{TV}(u)$$

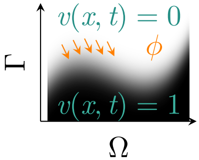


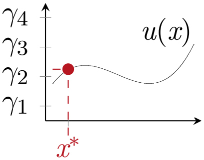
$k = 1$ | $k = 8$ | $k = 21$ | $k = 60$ | Input

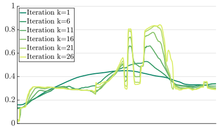
Implementation with prost library: Möllenhoff et al. 2016; Middlebury Stereo Datasets: Scharstein et al. 2014

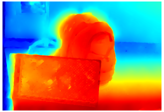
Conclusion

- Introduction of the lifted Bregman iteration
- Equivalence of original and lifted Bregman iteration if sufficient condition on subgradients is satisfied
- Sufficient condition satisfied in case of ROF problem with anisotropic TV
- First results of lifted Bregman iteration on originally non-convex problems









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