Recursive Green's Function Registration

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Abstract. Non-parametric image registration is still among the most challenging problems in both computer vision and medical imaging. Here, one tries to minimize a joint functional that is comprised of a similarity measure and a regularizer in order to obtain a reasonable displacement field that transforms one image to the other. A common way to solve this problem is to formulate a necessary condition for an optimizer, which in turn leads to a system of partial differential equations (PDEs). In general, the most time consuming part of the registration task is to find a numerical solution for such a system. In this paper, we present a generalized and efficient numerical scheme for solving such PDEs simply by applying 1-dimensional recursive filtering to the right hand side of the system based on the Green's function of the differential operator that corresponds to the chosen regularizer. So in the end we come up with a general linear algorithm. We present the associated Green's function for the diffusive and curvature regularizers and show how one may efficiently implement the whole process by using recursive filter approximation. Finally, we demonstrate the capability of the proposed method on realistic examples.

Key words: Nonparametric Image Registration, Green's Function, Recursive Filter

1 Introduction

The problem of image registration arises in many application of medical image processing and computer vision. Given two images, a reference R and a template T, we try to find a suitable transformation that aligns the template to the reference.

In this paper, we focus on non-parametric image registration, where we minimize a joint functional depending on a (dis)similarity measure and a regularizer. The optimization leads to a system of partial differential equations, the so-called Euler-Lagrange equations. In the literature one may find various ways for solving the PDEs, see [7] for an overview. In this paper, we introduce a generalization of solving this problem by convolution. For speed purposes, this convolution is approximated by 1-dimensional recursive filtering. The paper is organized as follows. We start by describing the registration problem in more detail followed by a short introduction to the Green's function approach. In particular we comment on an efficient way to determine the Green's function based on eigenfunction expansion. In the following we come up with a recursive filter approximation approach for the presented filters and present the resulting algorithm. Finally, we provide numerical examples for CT lung data.

2 Image Registration

In this section we will state the mathematical formulation for the non-parametric image registration problem and give an overview of well-known methods. For d-dimensional images $R, T : \mathbb{R}^d \to \mathbb{R}$, we are looking for a non-parametric transformation $\phi(\mathbf{x}) = \mathbf{x} - \mathbf{u}(\mathbf{x})$, where $\mathbf{u} : \mathbb{R}^d \to \mathbb{R}$ is such that the deformed template $T_u(\mathbf{x}) := T(\mathbf{x} - \mathbf{u}(\mathbf{x}))$ is similar to the reference R. In this context, the vector field $\mathbf{u}$ is called displacement field. This problem can be solved by minimizing the joint functional

$$\mathcal{J}[\mathbf{u}] = \mathcal{D}[R, T; \mathbf{u}] + \alpha \mathcal{S}[\mathbf{u}], \tag{1}$$

where $\mathcal{D}$ is a image similarity or distance measure and $\mathcal{S}$ is a regularization or smoothing term. The smoothing term constrains the transformation to a set of "reasonable" ones, which may be used to advantage for the given application. The regularization parameter α controls the relative contributions of the two terms.

2.1 Distance Measures

To measure similarity of the two images we have to define an appropriate distance measure $\mathcal{D}$. Frequently used measures are the sum of squared differences (SSD), normalized gradient fields [5] or mutual information [12]. In this paper, we will exemplarily use SSD, that is given by

$$\mathcal{D}^{\mathrm{SSD}}[R, T; \mathbf{u}] := \int_\Omega (T_u(\mathbf{x}) - R(\mathbf{x}))^2 \, \mathrm{d}x. \tag{2}$$

2.2 Regularizers

Basically regularizers measure the smoothness of the wanted the transformation. Popular regularization terms are defined as follows:

$$\mathcal{S}^{\mathrm{diff}}[\mathbf{u}] := \frac{1}{2} \sum_{j=1}^{d} \int_\Omega \|\nabla u_j(\mathbf{x})\|^2 \, d\mathbf{x}, \tag{3}$$

$$\mathcal{S}^{\mathrm{curv}}[\mathbf{u}] := \frac{1}{2} \sum_{i=1}^{d} \int_\Omega (\Delta u_l(\mathbf{x}))^2 \, d\mathbf{x}, \tag{4}$$

$$\mathcal{S}^{\mathrm{elas}}[\mathbf{u}] := \int_\Omega \frac{\mu}{4} \sum_{j,k=1}^{d} \left(\partial_{x_j} u_k + \partial_{x_k} u_j\right)^2 + \frac{\lambda}{2} (\nabla \cdot \mathbf{u})^2 \, d\mathbf{x}. \tag{5}$$

For fluid registration $\mathcal{S}^{\text{fluid}}[\mathbf{u}] := \mathcal{S}^{\text{elas}}[\mathbf{v}]$. In this context ∂_{x_j} denotes the partial derivative in direction x_j, ∇ the gradient, $\nabla\cdot$ the divergence and Δ the Laplacian operator. Furthermore $\mathbf{v} : \mathbb{R}^d \to \mathbb{R}^d$ is the velocity field and λ, μ are called Lamé constants. The regularization terms shown above are known as diffusive, curvature, elastic, and fluid registration, respectively (see [7]).

2.3 Numerical Solutions

For an optimal displacement field $\mathbf{u}$ the Gâteaux derivatives of the joint functional in Equation (1) vanishes. This leads to the corresponding set of non-linear Euler-Lagrange equations

$$\alpha\mathcal{A}[\mathbf{u}] = \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})), \tag{6}$$

where the differential operator $\mathcal{A}$ and the force $\mathbf{f}$ are the Gâteaux derivatives of $\mathcal{S}$ and $\mathcal{D}$, respectively (under the assumption of specific boundary conditions). The resulting differential operators for the various regularization terms (3), (4), and (5) are

$$\mathcal{A}^{\text{diff}} = \Delta, \tag{7}$$

$$\mathcal{A}^{\text{curv}} = \Delta^2 \text{ and} \tag{8}$$

$$\mathcal{A}^{\text{elas}} = \mu\Delta + (\lambda + \mu)\,\nabla\nabla\cdot\,. \tag{9}$$

The differential operator for fluid registration is similar to the elastic one, but operates on the velocity field as opposed to the displacement field as discussed in e.g. [7]. A fixed-point-type iteration scheme, such as

$$\mathcal{A}[\mathbf{u}^{k+1}] = \mathbf{f}(\mathbf{x}, \mathbf{u}^k(\mathbf{x})), \tag{10}$$

is a practical way to linearize and solve these equations. We have various options in providing a numerical scheme for first discretizing and solving the PDEs. The main work is the solution of the resulting linear system. This can be done by successive overrelaxation (SOR) [3], multigrid methods [6], additive operator splitting (AOS) for diffusive regularization and homogeneous Neumann boundary conditions [7] or by Fourier methods [2], to name few popular options. An alternative method, as suggested by Bro-Nielsen and Gramkow, for fluid registration [1] is to solve the Euler-Lagrange equations by means of the convolution operation

$$\mathbf{u}^{k+1} = G * \mathbf{f}^k, \tag{11}$$

where G is an appropriate filter kernel which turns out to be the Green's function w.r.t. the differential operator under the assumption of translational invariance. In the following we will assume this condition is fulfilled although boundary conditions are applied. This approach can be generalized to other regularizers as well.

3 Green's Function

The Green's function is used to solve linear inhomogeneous ordinary or partial differential equations, e.g. in electromagnetic theory [10]. We consider sufficiently smooth functions $\mathbf{u}, \mathbf{f}, \mathbf{g} : \mathbb{R}^d \to \mathbb{R}^d$, an open set $\Omega \subset \mathbb{R}^d$ with smooth boundary $\partial\Omega$ and the linear operator $\mathcal{A}$ for the Euler-Lagrange equations (6). Therefore, we have

$$\mathcal{A}\mathbf{G}^i(\mathbf{x};\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})\mathbf{e}_i \qquad \text{for } \mathbf{x} \in \Omega \tag{12}$$

$$\mathbf{G}^i(\mathbf{x};\mathbf{y}) = \mathbf{g}(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \partial\Omega, \tag{13}$$

where $\mathbf{G}^i = \left(G_1^i \dots G_d^i\right)^T$, is called *vector Green's function* in the direction $\mathbf{e}_i, i \in \{1, \dots, d\}$. Then $\mathbf{G}(\mathbf{x};\mathbf{y}) := \left(\mathbf{G}^1(\mathbf{x};\mathbf{y}) \dots \mathbf{G}^d(\mathbf{x};\mathbf{y})\right)$ is called *dyadic Green's function*, using which the following equation must hold:

$$\mathcal{A}[\mathbf{G}](\mathbf{x};\mathbf{y}) = \mathbf{I} \cdot \delta(\mathbf{x}-\mathbf{y}), \tag{14}$$

where $\mathbf{I} \in \mathbb{R}^{d\times d}$ is the identity.

3.1 Green's Function Calculation

To the best of our knowledge, there is no established and commonly accepted way to derive Green's function for a differential operator. Here, we will introduce the method of eigenfunction expansion.

Eigenfunction Expansion The idea of the eigenfunction expansion is to express the solution of a differential equation by a weighted sum of orthonormal eigenfunctions Φ_i, i.e.

$$u(\mathbf{x}) = \sum_{i=1}^{\infty} a_i \Phi_i(\mathbf{x}), \quad a_i = \frac{\Phi_i(\mathbf{y})}{\kappa_i}, \tag{15}$$

where κ_i is the eigenvalue that corresponds to the eigenfunction Φ_i (see e.g.[1]). Equation (15) leaves us with two problems. How to compute the eigenfunctions and eigenvalues for a given operator and how to deal with the infinite sum. For the latter problem an answer is also given in [1]. In the following, we present the eigenfunction expansions for diffusive and curvature registration for the 2-dimensional case. The extension to the 3-dimensional case is straightforward. The Green's function for the elastic regularizer coincide with the one of the fluid registration. The only difference is the fact that it is applied to the displacement instead of the velocity.

Diffusive Registration The Green's function for the diffusive operator $\mathcal{A}^{\text{diff}}$ under zero boundary conditions on the domain $\Omega =]0,1[^2$ is given by a scalar Green's function

$$G(\mathbf{x};\mathbf{y}) = -\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\Phi(\mathbf{y})}{\kappa^{k,l}} \Phi(\mathbf{x}). \tag{16}$$

with eigenfunctions and eigenvalues

$$\Phi^{k,l}(\mathbf{x}) = \sin(k\pi x_1)\sin(l\pi x_2) \quad \text{resp.} \quad \kappa^{k,l} = -\pi^2\left(k^2 + l^2\right). \tag{17}$$

One can shown that the diffusive operator is separable and therefore the displacement field can be computed by $u_1(\mathbf{x}) = (G * f_1)(\mathbf{x})$ and $u_2(\mathbf{x}) = (G * f_2)(\mathbf{x})$.

Curvature Registration As in the case of the diffusive registration, the operator for the curvature registration is separable as well. We also consider zero boundary conditions and the domain $\Omega =]0, 1[^2$. The eigenfunctions and eigenvalues are given by

$$\Phi^{k,l}(\mathbf{x}) = \sin(k\pi x_1)\sin(l\pi x_2) \quad \text{resp.} \quad \kappa^{k,l} = \pi^4\left(k^2 + l^2\right)^2. \tag{18}$$

3.2 Deriving a Recursive Filter

In this work the Green's function $G(\mathbf{x};\mathbf{y})$ is given in its continuous eigenfunction representation. The force field is given by Equation 2. Both of them are discretized on a finite grid. We get the Green's filter $\mathbf{G}$ and the discrete force field $\mathbf{F}$. The discrete displacement field $\mathbf{U}$ is now given by the convolution $\mathbf{U} = \mathbf{G} * \mathbf{F}$. This operation has high computational costs, i.e. $O(N^2)$, where N is the number of voxel. To reduce these costs the idea of 1-dimensional recursive filtering will be introduced. In the following we will compute a separable filter approximation where the recursive filtering in each direction can be handled step by step. This allows us to use an implementation of diffusive and curvature registration with the computational costs of the demons approach using a recursive implementation of the Gaussian.

We start with the 2-dimensional case. In terms of computing a separable recursive filter approximation for the Green's filter $\mathbf{G} \in \mathbb{R}^{n_1 \times n_2}$, we need a separable approximation $\widetilde{\mathbf{G}} \in \mathbb{R}^{n_1 \times n_2}$ of $\mathbf{G}$ of the form

$$\widetilde{\mathbf{G}} = \mathbf{x} \otimes \mathbf{y}, \quad \text{resp.} \quad \widetilde{\mathbf{G}} = \mathbf{x}\mathbf{y}^T, \tag{19}$$

where $\mathbf{x} \in \mathbb{R}^{n_1}$ and $\mathbf{y} \in \mathbb{R}^{n_2}$. This is the case if $\operatorname{rank}(\widetilde{\mathbf{G}}) = 1$. We choose this approximation to be optimal in terms of the Frobenius norm ($\|\cdot\|_F$) that computations are easy to handle. This optimal approximation is called a rank-one approximation and can be achieved by a singular value decomposition (SVD)[4]. A singular value decomposition is possible for all matrices $\mathbf{G}$. Let $\mathbf{G} = \mathbf{U}\mathbf{S}\mathbf{V}^T$, where $\mathbf{U} = \left(\mathbf{u_1} \dots \mathbf{u_{n_1}}\right) \in \mathbb{R}^{n_1 \times n_1}$ and $\mathbf{V} = \left(\mathbf{v_1} \dots \mathbf{v_{n_2}}\right) \in \mathbb{R}^{n_2 \times n_2}$ are orthogonal matrices and $\mathbf{S} \in \mathbb{R}^{n_1 \times n_2}$ is a diagonal matrix. Then $\widetilde{\mathbf{G}} = \mathbf{u}\mathbf{v}^T = \left(\sqrt{s_{11}}\mathbf{u}_1)\right)\left(\sqrt{s_{11}}\mathbf{v}_1)\right)^T$. For the 3-dimensional case the application of the SVD is not possible in this way. A rank-one approximation of a third order tensor can be seen as a generalization of the SVD and is often called multidimensional SVD. This problem can be attacked by the generalized Rayleigh-Newton iteration as described by Zhang and Golub[13]. The separable approximation $\widetilde{\mathbf{G}}$ is then given by $\widetilde{\mathbf{G}} = \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$.

The d-dimensional convolution can now be substituted by d 1-dimensional convolutions what is already a nice speed-up. But the 1-dimensional convolution has still high costs. That is where recursive filtering becomes an issue. The recursive filter approximation of the 1-dimensional filters is done by using Matlab implementation of Prony's method [8] with a zero padding to emphasize the zero-boundary conditions in the filter. It should be mentioned that the combination of causal and anti-causal recursive filter approximation does not lead to a perfect symmetry in this implementation. However, in our case this aspect is hardly noticeable. This recursive filtering scheme can now be implemented with O(N) computation cost.

4 Numerical Examples

In the following evaluate the capability of the Green's function registration on the POPI breathing thorax model of the Léon Bérard Cancer Center & CREATIS lab, Lyon, France [11]. Before starting the evaluation we leave some words on the algorithm. We modified the fixed-point type iteration scheme (Equation (10)). Instead of computing the whole displacement field in one iteration, we compute update steps $\mathbf{u}^k$ by recursive filtering the force field. The final displacement field after n iteration steps is then given by $\mathbf{u} = \sum_{k=1}^{n} \mathbf{u}^k$. The modification is already mentioned in [7] for elastic registration. To have more control over the iteration step we included an Armijo line search for these updates. To allow larger deformations we start with an affine-linear preregistration followed by a multi-level Green's function registration. The choice of the distance measure is the SSD (Equation 2). This is not the best choice for the registration of lung volumes but the aim of this paper is to show the capability of the proposed linear algorithm. Stopping criteria on each level is a difficult topic. Here, we consider the relative distance measure $\bar{d}_k = \mathcal{D}^{\text{SSD}}(R, T; \mathbf{u}^k)/\mathcal{D}^{\text{SSD}}(R, T; \mathbf{0})$. The algorithm stops when the change of the relative distance $\bar{d}$ is less than 1%, i.e. $\bar{d}_k - \bar{d}_{k-1} < 0.01$. Furthermore a maximum number of iterations is given that is descending as the level is ascending.

The POPI-model provides a 4D-CT series of the lung, including ten 3D-CT volumes $(v_0, \ldots, v_9)$ representing ten different phases of an average breathing cycle. For evaluation purposes a set of corresponding landmarks is given for all these volumes. For the registration the preprocessed versions of these volumes with removed background and reduced image size are used. This results in an image size of 482 x 360 x 141 with resolution 0.976562mm $\times$ 0.976562mm $\times$ 2mm. Our registration works on an image size of 256 $\times$ 256 $\times$ 128 computed by linear interpolation. Therefore the resulting working resolution on the finest level is 1.8387mm $\times$ 1.3733mm $\times$ 2.2031mm. We perform a curvature and a diffusive recursive Green's function registration on these data. An exemplarily result for the diffusive registration is shown in Figure 2. To coincide with the results presented with the model computed with a demon-based approach, we use the second volume v_1 as the reference and all others as templates. For each registration the mean and standard deviation of the 40 landmarks are given for the initial situation as well as the mean and standard deviation of the TRE. Additionally the maximal TRE is presented (see Figure 1). Considering the

Fig. 1. Results of the recursive curvature Green's function registration on the POPI-model in mm. The row with the registration index v_1v_2 presents the registration result of the registration of volume v_1 and volume v_2. Presented are the mean and standard deviation of the initial landmarks (second column) and the TRE of the registration presented with the POPI-model (pTRE, third column), the curvature registration (cTRE, fourth column) and diffusive registration (dTRE, fifth column). The last three columns show the maximal TREs for POPI, curvature and diffusive

Reg	Data μ/σ	pTRE μ/σ	cTRE μ/σ	dTRE μ/σ	pMax	cMax	dMax
10	0.48 / 0.54	1.3 / 0.3	0.67 / 0.35	0.68 / 0.34	1.8	1.79	1.75
12	0.49 / 0.62	1.4 / 0.2	0.71 / 0.37	0.69 / 0.42	2.1	1.69	1.57
13	2.19 / 1.82	1.4 / 0.4	1.58 / 0.89	1.33 / 0.70	2.3	3.63	2.70
14	4.33 / 2.51	1.2 / 0.4	1.42 / 0.83	1.20 / 0.73	2.3	5.03	4.32
15	5.75 / 2.64	1.3 / 0.5	1.50 / 0.99	1.42 / 0.92	2.6	6.27	5.48
16	6.10 / 2.92	1.1 / 0.4	1.43 / 0.84	1.30 / 0.72	2.0	5.29	4.20
17	5.03 / 2.33	1.3 / 0.5	1.42 / 0.88	1.34 / 0.76	2.4	4.66	3.56
18	3.68 / 1.57	1.1 / 0.3	1.12 / 0.80	1.12 / 0.72	1.7	3.76	3.71
19	2.07 / 1.06	1.1 / 0.3	0.95 / 0.68	0.88 / 0.58	1.9	3.18	2.97
Total	3.35 / 1.78	1.2 / 0.4	1.20 / 0.74	1.17 / 0.66	2.6	6.27	5.55

working resolution these results are very satisfying. The mean μ of the TRE of the three methods is nearly the same. However, the standard deviation σ recursive Green's function registration is larger. This might have two reasons. Firstly, the errors occur mainly in the x_3-component where we use a lower resolution. The other and probably more important reason results from the used distance measure. As mentioned before we apply the SSD without further preprocessing to take the lung density change into account as proposed in [9]. To achieve better results, a next step would be to incorporate this preprocessing or to use a more suitable distance measure for this specific application. This would also help to catch the outliers. Another important information from the table is the fact that the curvature Green's function registration produces less accurate but smoother results than the diffusive one, so the behavior of the regularizers are preserved in this method.

5 Conclusions

This papers presents a generalization of the registration by a convolution with the Green's function. It was shown that also for the diffusive and curvature regularization filter kernels can be established and recursively approximated. This helps us to speed up the computation in such a way a linear algorithm could be presented. The numerical section presents promising results of this approach by registering a set of CT lung data. Next steps include e.g. the search

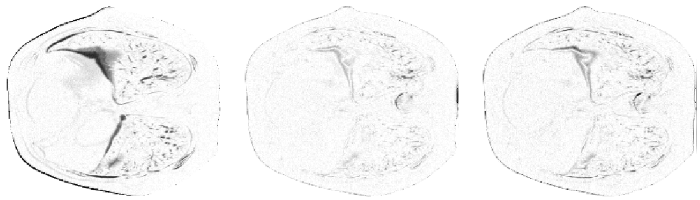

Fig. 2. The absolute difference of the template and the reference before (left) and after the diffusive (middle) and curvature (right) recursive Green's function registration for an exemplarily slice

for the best filter kernel size on each level and the use of application specific distance measures.

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